The Cybernetic Theory of Development

Mathematical Models for A Re-Evaluation of the Is-Ought Problem

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Introduction to the Problem of 'Is' and 'Ought'

One can hardly deny that the absolute separation of 'is' and 'ought' by Hume and Kant for a good part was due to their willingness to conciliate between God and human reason. Wittgenstein in his own separation of 'ought' from 'is' was quite outspoken. He writes "God does not appear *in* the world" (Tractatus, 6.432) meaning that "Ethics is transcendental" (ibid., 6.421). To show this was Wittgenstein's very aim: "The aim of my book is ethical, . . . My book confines the ethical, as it were, from the inside and I am convinced that only so it is exactly confinable" (in a letter to Ludwig von Ficker).

Wittgenstein's 'Tractatus' is the work which perhaps more than any other single accomplishment underlies the linguistic separation of 'is' and 'ought' in modern logic. It can be examined as an effort

1° to build a language in sentences of which all meaningful thoughts concerning 'is' could be voiced and

2° to show that all ethics, i.e. thoughts concerning 'ought', are so excluded.

Following up program 1 Wittgenstein constructed the simple prototype of what is now called an *extensional language*. Later, even languages directly or indirectly concerning 'ought' have become objects of descriptive study, leading to constructions of formalised *intentional languages* in deontic logic and in modal logic in general.

On which grounds lies the separation of 'is' and 'ought' in its modern, linguistic form?

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Let us approach the problem in terms of an illustrative model. Let X be the set of the thinkable states of the world. Then the set F(X) of all the subsets of X is the set of all the thinkable states of affairs. For

instance, the state of affairs that there is a red flower in a certain vase is represented by a certain subset A of X. This subset comprises all the states of the world in which the vase in question has a red flower. In effect every local state of affairs, local either spatially or temporally, singles out a proper subset of X, for it does not distinguish between the properties of the states of the world outside of some local environment.

To each thinkable state of affairs $A \subset X$ there is a meaningful sentence p_A in our extensional language, expressing that this state of affairs is valid. The sentence p_A is true if the real state of the world $\hat{x}$ belongs to A, otherwise false. The logical negation, conjunction, and disjunction of sentences correspond to the formation of complement, cut, and union of the respective sets. This way we can build sentences also syntactically. All the sentences expressing the state of affairs X are logical ('analytical') truths. Other true sentences express empirical ('synthetic') truths or facts.

So we have characterized a logic of 'is' given in an extensional language (at an empirical level).

But every real person is capable of distinguishing only a finite number of states of affairs from one another. Then for each person there is a partition B_i of X, such that this person is able to grasp only the states of affairs representable as unions of the subsets B_i. These states of affairs together make his personal, local world. We can say that the person is 'conscious' of these states of affairs only.

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Let us extend the model to grasp some aspects of 'ought' too. Every *act* certainly can be represented as a function f from X to F(X). This says only that the state of the world x, in which the act is performed, together with the act f itself singles out a definite state of affairs $A \in F(X)$ as the 'consequence' f(x) of the act f under the premise x: $f(x) = A$. What peculiar state of affairs A is so picked depends on the definition of the act f in question as an empirical construct.

For instance, consider the act of closing a window. If we regard the act completed first after the window has been actually closed, we have a constant act f whose consequence f(x) is the same state of affairs $C \in F(X)$ for any state $x \in X$. But if we consider as act the movements undertaken in order to close the window, then the act may either succeed or fail. In this case we have two possible consequences, viz. the state of affairs C and its complement $\overline{C}$ in X.

When speaking of acts we usually mean the *conscious acts* of some person or community. This is an act whose possible consequences are states of affairs which are conscious to the actor in question. A conscious act thus is a function f from X to F(X) such that the possible values f(x) of it are some subsets $C_1, C_2, \ldots, C_n$ of X, finite in number, and representing some unions of the respective partition sets B_i. The actor knows only the possible consequences $C_1, C_2, \ldots, C_n$ but not usually the state of the world in which he is acting. That is to say he is usually acting under the conditions of uncertainty.

The notions of act and conscious act so defined include, as far as I can see, all the meanings in which one has spoken of acts. We have stated only that an act may bring into being some state(s) of affairs. This I think cannot be denied of any kind of act.

The notion of value-judgement can in equally general terms be defined as setting up a certain preference order between the possible consequences of the possible acts an actor is capable to conceive. Every subjective value-judgement of a real person is based on a certain preference order of some states of affairs conscious to him, i.e. of his B_i's or their unions. If there is an objective ethical value, it must be represented by an order of preference of the states of the world $x \in X$.

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Admittedly all our *observations* are acts which bring into our consciousness a certain empirical fact, viz. that the true state of the world $\hat{x}$ belongs to a certain proper subset A of X or, what is the same, that a certain state of affairs A is valid — and nothing more. This is what Wittgenstein obviously wanted to say. (An observation thus is a function f from X to F(X) such that $x \in f(x) = A$ for any x and A.)

But is all ethics really so excluded from any confrontation with empirical knowledge, indeed outside the capacity of human reason ("transcendental"), as Wittgenstein thought?

In practical life we of course set up many kinds of orders of preference between different states of affairs, say, to settle which instrument is best for each task, and all this is done on the basis of the observations we have on the relevant properties of the different instruments. However, a genuine ethical value is concerned only when it comes to the meaning of life or perhaps of the world in general. Wittgenstein stated his point as follows.

"The meaning of the world must be outside of the world. In the world

everything is as it is and everything happens as it happens . . . All what happens and is in this or that way is chance. What makes of value something else than chance cannot be of this world, it cannot be *in* the world, since otherwise it would itself be chance. It must be outside of the world;" (Tractatus, 6.41.)

Wittgenstein seems here to voice the idea that a genuine ethical value is connected with some kind of *necessity*, a factual necessity and not merely a logical one. As he admits only logical necessity as existent in the world (sentence 6.375), he cannot find any genuine value *in* the world.

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But if you take as the 'world' the *human world*, composed of mankind and its productive system (inclusive the relevant part of nature), you definitely have better prospects to find a factual necessity and also a genuine ethical value in the world. The natural laws of course restrict the factual possibilities of action of human beings. And so do all the previous, completed acts of people. Natural laws + previous human activity together determine the factual necessity in the framework of which every human actor whether an individual or a collective is bound to act.

Translated to the language of our model: If X is interpreted as the set of the thinkable developmental states of the human system, then there is a proper subset D of X which expresses the factual necessity, and comprises the *possible states* of the system. Possible acts are those whose possible consequences are composed solely of the elements of D.

Further on: If X_t and D_t mark the respective sets of states at the moment t of a (for the sake of simplicity) discrete calendar, then D_t is determined by the developmental law of the human system. This law expresses the natural laws + the laws due to previous human action. Such a law is representable by a function φ from X_t to X_{t+1}, under the assumptions made.

Now we can make the crucial question. Is there such a developmental law governing the history of mankind? A linguistic idealist is inclined to call such a law a "metaphysical belief" But for a cybernetician the question is quite meaningful. He would answer that the answer wholly depends on the structure of the system in question. If there are suitable couplings of feedback in the structure of the human system, then it is quite possible that a law φ exists, and gives to the system a

goal-directed, "purposive" nature. He would call D_t the *domain of ergodicity* or the domain of selfsteering of the system at the moment t.

The law of development φ expresses causality and thus is deterministic. But human beings as active parts of the system are themselves creating a share of this causality. So the development is not necessarily blind but is capable of expressing also human intentions.

The theoretical limits within which the human pursuit may express itself are determined by the domain of ergodicity, and thus ultimately by the natural laws + earlier human history. The largeness of this domain thus is a measure of the capability of human beings to realize their humanity — or whatever they want to realize — by means of their power over the natural forces and over the laws of their own development. In the sense first defined I think by Engels it thus is a measure of human progress and indeed an objective foundation of ethical value — as derived from his world.

To sum up: in a cybernetic construction, taking into account the feedback structure of the human system, you may get a more intricate relation between 'is' and 'ought' than is customary assumed in current linguistic construction. This is the idea we are going to study in this book.

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How much Marxist is this idea? The work of Oskar Lange and Marxist philosophy will be much quoted in the book, the latter especially in the form exposed by the GDR collective of philosophers in their book 'Marxistische Philosophie' (Dietz, Berlin 1967). But the present book only suggests mathematical models, and such models must be distinguished from philosophical formulations at the verbal level. So it is more exact not to call my approach a Marxist one but rather — cybernetic.

Because of the conspicuous similarity of some Marxist formulations and certain cybernetic ideas Marxist terminology will often accompaign the cybernetic constructions in this book. But here we of course meet the problem of compatibility of verbal language and mathematics. To my mind a mathematical model is in science always something more than a verbal philosophical formulation. It at least attempts at an unambiguity and exactness which is unattainable when using verbal language only. Therefore the cybernetic constructions in this book are an autonomous theory, which must be appraised on its own accord only.

The aim of the book is thus a cybernetic one: to introduce a cybernetic approach to the problem of 'is' and 'ought'. Who wants to grasp the idea quickly he can read Chapters III and V. For a more complete picture include Chapter II and Chapter IV. Chapter I is there only to make a textbook: to give the necessary details for a beginner who is not privy to elementary mathematics. Chapter I is quite trivial, but to my experience just this kind of knowledge a social scientist wanting to read cybernetics is wanting.

Contents

CHAPTER I:

Elementary Mathematical Tools

1 § What Is Mathematics?

1 / A Formal Language?

The most striking thing in mathematics undoubtedly is its playing with signs. Sometimes mathematics itself was held a conventional game comparable to chess.

To have a more realistic picture of mathematics we should begin by distinguishing between an empirical and a theoretical level of knowledge. Our sensory observations concern what could be called 'empirical objects'. The empirical objects are objects which — though they are themselves idealisations of a certain theory conceived in our mind — are directly related to the real objects, and represent the images of the latter in our consciousness.

The theoretical objects dealed with at the theoretical level of knowledge are usually not directly related to the observations of real objects. The human mind 'invents' them by her creative imagination, out of the material offered by the observations of reality. The theoretical constructs considered in mathematics are no exception to this general rule. ***In this sense we can say that the mathematical notions are based on the observation of reality.*** And their final purpose is like the purpose of any theoretical constructs to 'explain', to give systematics to observations.

So the playing with signs in mathematics is not an end for itself. Still the game-aspect too is there. It is inherent in the study of mathematics as a deductive system, as a *formal language* (a *calculus*). In terms of elementary mathematical notions — you find them explained later in this chapter — a formal language can be characterized as follows.

It has a finite or infinite set E of basic signs. Every finite or infinite sequence zuv . . . of the basic signs belongs to the set M of signs considered in the language. There is a subset A of M comprehending the 'expressions' of the language. There is a function f from the set F(A) of all the subsets of A to the same set F(A), such that

1° $X \subset f(X)$ for any $X \subset A$,

2° $f(X_1) \subset f(X_2)$ for any $X_1 \subset X_2$,

3° $f(f(X)) \subset f(X)$ for any $X \subset A$,

4° for any $z \in f(X)$ and any $X \subset A$ there is a finite set $X^* \subset X$ such that $z \in f(X^*)$, and

5° there is a set $S \subset A$ such that $f(S) \subset S$.

Here S is the set of the 'theorems', or true sentences. The function f is the operation of logical deduction as defined for this language. The study of the formal languages, i.e. combinations (E, M, A, f), involved in mathematics means studying the *logical syntax* of mathematics.

In the *logical semantics* of mathematical notions one studies the connections of formal languages with the theoretical objects of which is spoken in the language. A formalisation of the language used in a semantic analysis leads to the notion of 'meta-language', i.e. a language used when studying the logical properties of the object language. This evokes the question on the existence of a single great formal language in terms of which all mathematics could be expressed. Is that possible?

2 / The Failure of the Formalist Program

If mathematics is a formal language, it should be possible to 'play mathematics' without taking any notice of the meanings associated with mathematical symbols. The mathematical truths could be derivable by purely formal rules of mathematical calculus, assuming one could but find out the rules of this game. How should such a calculus look like?

One comes to think of a calculus, where all the true statements can

be formally derived from a number of axioms by a number of rules of inference. By means of the rules of inference a complete recursion of all the mathematical truths to the chosen axioms would be possible. Therefore, this hypothetical calculus was called the 'Complete Recursive Calculus' by the Formalists who had the idea in mind.

The idea of CRC was refuted by a well known theorem of Gödel (Kurt Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, *Monats. Math. Phys.* 38, p. 173—198, 1931). The theorem states that any adequate consistent arithmetical calculus is incomplete: there are always true statements about the integers which cannot be proved within the calculus, that is, which cannot be derived from the axioms of the calculus by the rules of inference of the calculus whatever axioms and whatever rules of inference we might assume. This means that a class of fundamental mathematical calculi are rather IRC, Incomplete Recursive Calculi, than CRC.

What is the philosophical content of Gödel's proof? It gives evidence to the effect that mathematics cannot be comprehended as a closed, formal, and conventional system as was supposed by the Formalists. On the other hand, Gödel's theorem is in good accord with dialectical materialism, according to which even the mathematical truths reflect material reality. The component of reflexion, the foundation of mathematical truths on the reflexion of material reality in the consciousness of the human being, cannot be eliminated from mathematics just as little as it can be eliminated from physics, from botanics, or from zoology. Mathematics, just like physics, or botanics, or zoology is a science telling something on the phenomena of material reality. It dwells on different phenomena and on a higher level of generality than the other sciences but is still as deeply anchored in our observation of material reality as are the other sciences. The use of formal symbols helps in the communication of mathematical ideas but mathematics is not just a formalism.

Though the early Formalist Program was wrecked by Gödel's theorem, the conception of mathematics as a formal language is today almost as strong as it was in 1931 — and before that — in the countries of Positivism. This is because this conception is a part of the positivistic interpretation of sciences in terms of linguistic idealism. "Logic and mathematics are based, according to the ideas of the neopositivists, on a system of quite arbitrarily assumed axioms and rules, which is a similar product of convention as are the rules of chess and a card game." (*The Foundations of Marxism-Leninism*, 2. Finnish edition,

p. 47—48, 1961). Indeed, the conception of mathematics as a formal language composed of axioms and rules of inference chosen by more or less aesthetic convention ("simplicity" etc). goes on as a part of the positivistic philosophy of mathematics. Another part of this philosophy is the conception that all that is fundamental in mathematics can be attained at by a logistic analysis of the logical syntax of the 'language of mathematics'. This is wrong too: there are obvious non-linguistic aspects of mathematics.

3 / Non-Linguistic Aspects of Mathematics

Every mathematician knows by experience what a great role sensations, perceptions, visual images and geometrical illustrations play in mathematical thinking. All these are elements that connect the mathematician's work with the observation of material reality and with the so-called 'empirical' sciences .A successful mathematician is in his work often directly pulled by concrete problems of physics, chemistry, biology, or economics. The aspects of mathematics as a science studying problems of material reality are badly neglected by the purely linguistic conception of mathematics.

A more realistic philosophical interpretation of mathematics must start with the fact that mathematics — like physics, botanics, or zoology — is a science devoted to the study of certain aspects of material reality. Physical particles, plants, or animals are not themselves studied by mathematics but certain sets of particles, animals, and plants, and certain proportions holding between the elements of such sets may well be. Similar sets and similar proportions may appear among other things of reality. Mathematics is a science studying such sets and proportions whereever they may appear in reality, or in an imaginable reality.

A student of mathematics is soon fascinated by the generality of the aspects of reality revealed to mathematical observation. The sets and the proportions studied in mathematics approach universal validity, and appear in the most different contexts when compared with the more restricted aspects of reality studied by physics, botanics, or zoology. Still one can say that the starting point of mathematical study is in the observation of certain *specific proportions* holding between the elements of certain specific sets. Mathematical discovery is based rather on the study of the specific and concrete than of the general and abstract. To solve the seeming paradox we here meet: the generality is not pri-

marily sought in mathematics: the generality comes from the fact that a closer examination of just the exact, specific proportions holding between the elements of a certain set often reveals similar specific proportions as appear in some other sets of quite a different nature. It is a typical characteristic of the mathematical method, I dare to say, that it is based on a close examination of the specific prior to the general. Here again is a non-linguistic feature of mathematics which is missed by the positivist philosopher who is interpreting the *whole* mathematics in terms of the general syntactical categories of a language.

Above I have referred to some of the non-linguistic aspects of mathematics which the positivistic linguistic interpretation of mathematics fails to see. On the other hand, they are well in accord with the reflexion theory of dialectical materialism. To make it more precise and sum up the materialistic point of view we can say:

Logical deduction based on some axioms and on some rules of inference is an aspect of mathematical reasoning. However, in addition to the linguistic, grammatical element associated with the language of deduction there are non-linguistic, non-grammatical aspects of mathematics. Mathematical truth is not based on the finding of a 'correct' logical syntax for the language composed of the axioms and of the rules of inference but rather on the correct observation of certain specific proportions holding between elements of reality. Thus the foundation of mathematics is not to be sought solely in its linguistic grammary but in the whole reflexion of reality in human consciousness, in the observation of reality. This observation never ceases to bring forth new features of reality, and thus new features of mathematics.

Of course, this does not mean diminishing the value of the linguistic, logistic study of mathematical rules, but it only rejects a philosophical interpretation of mathematics, based on solely its linguistic aspects.

2 § Elementary Mathematical Notions

1 / Set

We spoke above of the observation of specific, exact proportions holding between the elements of some sets. These sets and these proportions, with which mathematics is concerned, are something existing in reality, or represent something which may exist in reality. What are the "elements" and what are the "sets"?

If we consider the material components of which, say, a motor car is composed, these components together give us an example of what is meant by a set. However, these components form the same set irrespective of whether we consider them as members in the whole called 'car' or whether we consider them as separated from one another, i.e. if we take the car to pieces and consider the collection of these separated pieces. Thus the motor car is something more than just a set of its elements: it is a set where the elements stand in particular proportions to one another. A material thing is a whole where both a set and some proportions holding between the elements of this set are involved. Distinguishing, when observing a material thing, between the set of its components and the proportions of these components (elements) to one another is the beginning of mathematics. The fact that a component x is an element of the set A of all the components is denoted as an "epsilon relation" as follows:

$$x \in A.$$

A property of the ordinary material components of which a material thing is composed is that they can be ordered to form a sequence. We can, for instance, order the components of a motor car along a straight line where each of the components lies between two neighboring ones. We can then associate a number to indicate the order of the components in the sequence, and write the sequence as follows: $x_1, x_2, \ldots, x_n$, where each of the x_k represents one component. If A is the set composed of these components we now denote the set by writing

$$A = \{x_1, x_2, \ldots, x_n\}.$$

Another way of denoting the set A is to introduce its "general element" and write

$$A = \{x;\ x = x_1, x_2, \ldots, x_n\}.$$

In this denotation the $x_1, x_2, \ldots, x_n$ are the "values" allowed for the general element in the set A. For each element we can write epsilon relation separately: $x_k \in A$ for each $k = 1,2, \ldots, n$.

Obviously the conception of natural numbers, by which we mean positive integers, is closely connected with the conception of the set of the material components of a material thing. Once a human being has been able to distinguish between such a set and the proportions in which the elements of this set stand to one another he has began

to develop the conception of natural numbers. Thus the beginning of mathematics has been simultaneously the beginning of counting things.

If we take first a material thing to the pieces $a_1, a_2, \ldots, a_n$, and then each of these components to smaller pieces, and continue in this way, it may happen that the reduction can be continued without any limit. We do not know if this is really possible — even the modern theory of elementary particles gives no definite answer to this question. However, to the possibility of such an unlimited reduction of a material thing there corresponds the idea of continuing the sequence of natural numbers without limit: 1,2,3, . . . The number of the elements of the set $A = \{1,2,3, \ldots\}$ is greater than any fixed number n. We call such a number infinity and denote it by the symbol ∞. The set of all natural numbers can now be denoted by $A = \{1,2, \ldots, \infty\}$ or, using the general element, by $A = \{x\,;\, x = 1,2, \ldots, \infty\}$.

With the acceptance of sets having an infinite number of elements some strange phenomena appear. Take, for instance, the set

$$B = \left\{ \frac{m}{n}\,;\, m = 0, \pm 1, \pm 2, \ldots, \pm\infty\,;\, n = +1, +2, \ldots, +\infty \right\}$$

of all rational numbers. At first glance it seems that this set has more elements than the set of natural numbers, since all the natural numbers are included but there are, in addition to this, all the non-integer rational numbers, and zero. However, we can order the rational numbers beginning with:

$$0, \frac{1}{1}, -\frac{1}{1}, \quad \frac{1}{2}, -\frac{1}{2}, \quad \frac{2}{1}, -\frac{2}{1}, \quad \ldots$$

Then we can give each rational number in this sequence an order number writing this sequence again as

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, \ldots$$

Thus we observe that the set B of all rational numbers and the set A of all natural numbers must be understood to have the same "number of elements" in the sense that both of these numbers can be ordered to an infinite sequence of numbers. We express this by saying that the sets A and B have the same *cardinal number* $\aleph_0$ (the letter reads "alef"). By the cardinal number of a finite set we understand the number n of its elements. We write $\aleph_0 > n$.

To express the fact that all natural numbers are among the rational numbers we can use the "inclusion relation" by writing

$$A \subset B.$$

A is called a *subset* of B. If A and B are whatever two sets of whatever elements, then $A \subset B$ if, and only if all the elements of A are also elements of B. In other words, we have an inclusion relation $A \subset B$ if, and only if from an epsilon relation $x \varepsilon A$ follows another epsilon relation $x \varepsilon B$ for every element x of A. Using the symbol of "implication" $\Rightarrow$ for the phrase "it follows" we can say that $A \subset B$ means the same as $x \varepsilon A \Rightarrow x \varepsilon B$.

A subset A of a set B which is different from B is called a *proper subset* of B. We have observed that even a proper subset A of B may have the same number of elements as the main set B, if B is an infinite set. For a finite set this is obviously not possible.

The set composed of all the subsets of a set B is denoted by $F(B)$. The set B itself is counted as an element of $F(B)$, and so is the "empty set" Φ that contains no elements; $B \varepsilon F(B)$ and $\Phi \varepsilon F(B)$. From each element x of the set B we get the subset $\{x\}$ containing only the element x; $\{x\} \varepsilon F(B)$ or, what is the same, $\{x\} \subset B$.

For each element $x \varepsilon B$ and $A \varepsilon F(B)$ we have either $x \varepsilon A$ or $x \bar{\varepsilon} A$ (x does not belong to A). Thus we have two possible relations for each element of B and each subset of B. Going through all the elements and subsets of a finite set B we get all in all 2^n possible combinations of elements of B into subsets of B, provided that n is the finite number of elements of B. Thus the number of subsets of B is 2^n. Indicating the number of elements of a set by the sign $\#$ we thus have, for any finite n,

$$\#B = n, \quad \#F(B) = 2^n \quad \text{(a finite } n\text{)}.$$

This rule can be extended to infinite sets. If $\mathfrak{m}$ is the cardinal number of the infinite set B, then the cardinal number of $F(B)$ is $2^{\mathfrak{m}}$, there being always $2^{\mathfrak{m}} > \mathfrak{m}$;

$$\#B = \mathfrak{m}, \quad \#F(B) = 2^{\mathfrak{m}} > \mathfrak{m} \quad \text{(an infinite } \mathfrak{m}\text{)}.$$

Accordingly, there is an infinite sequence of distinct infinite cardinal numbers, since we can begin with $\aleph_0$, and generate the infinite sequence

$$\aleph_0, \quad 2^{\aleph_0} = \aleph > \aleph_0, \quad 2^{\aleph} = \aleph_1 > \aleph, \quad 2^{\aleph_1} = \aleph_2 > \aleph_1, \qquad \text{etc.}$$

An example of an infinite set of the cardinality $\aleph$ is the set R of all

real numbers. We get all the real numbers by taking first all the rational numbers of the form $r = \frac{m}{n}$ (m and n integers), and then add the limits $r = \lim \frac{m}{n}$ where m and n are integers approaching infinity. The latter kind of real numbers are called irrational numbers. One cannot order all the real numbers to a single sequence. Thus the cardinality of R exceeds the cardinality $\aleph_0$ of the natural numbers and of the rational numbers. One can show that this cardinality is equal to $2^{\aleph_0}$. It is denoted by $\aleph$, as was indicated above, and called the cardinality of the 'continuum'.

In a closer examination of infinite sets we meet surprising curiosities and paradoxes, which have became a common object of study of higher set theory and mathematical logic. Here we do not go into these problematics but finish our introduction to the notion of set by introducing some notations useful when operating with sets.

If A and B are two arbitrary sets we can form their *union* $A \cup B$, their *intersection* $A \cap B$, and their cartesian (or set-theoretical) *product* $A \times B$. These new sets are defined by

$$A \cup B = \{x\,;\, x \in A \text{ or/and } x \in B\},$$
$$A \cap B = \{x\,;\, x \in A \text{ and } x \in B\},$$
$$A \times B = \{(x, y);\, x \in A \text{ and } y \in B\}.$$

The union $A \cup B$ thus is the set of all the elements who belong either to A or to B or to both of them. The intersection $A \cap B$ is the set of the common elements of A and B. The product $A \times B$ is the set of all the pairs (x, y) where x belongs to A and y to B.

All these operations can be generalized to any number of sets:

$$A_1 \cup A_2 \cup \ldots \cup A_n = \{x;\, x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n\},$$
$$A_1 \cap A_2 \cap \ldots \cap A_n = \{x;\, x \in A_1 \text{ and } x \in A_2 \text{ and } \ldots \text{ and } x \in A_n\},$$
$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n);\, x_1 \in A_1, x_2 \in A_2, \ldots x_n \in A_n\}$$

2 / Function

Another important notion of mathematics is connected with the specific proportions in which the elements of sets may be with one another. This is the notion of function.

If we associate with each element x of a set A one and only one element y of another set B we have defined a *function* f from the set A to the set B. This is denoted by writing

$$f: A \to B.$$

A function f thus arranges to every element x of A an element y of B in such a way that the element y is uniquely determined by the element x. The element y is called the "value" of the function f corresponding to the "value" of the argument x. The connection between the value of the function and the value of the argument is expressed by writing

$$y = f(x), \quad \text{or} \quad x \xrightarrow{f} y.$$

The set A of all the arguments of f is called the domain of definition of the function f, and is denoted by D_f. Thus $D_f = A$. The set of all the values of the function f is called the range of the function f, and is denoted by R_f. Thus $R_f = \{f(x); x \in A\}$. The range is obviously a subset of B: $R_f \subset B$. It may happen that R_f is a proper subset: $R_f \neq B$. This means that not every element of B appears as a value of the function f. In this case we can say that f is a function from A *into* the set B. If $R_f = B$, we can say that f is a function from A *onto* the set B.

There may be several elements $x_1, x_2, \ldots$ of A which all have the same element y of B as the value of the function f: $f(x_1) = f(x_2) = \ldots = y$. Thus we observe that to an element y of R_f there may correspond, in the association committed by the function f, several elements of A so that the functional relation between x-elements and y-elements cannot be inverted.

However, if the function f is such that to every two different elements x_1 and x_2 of A there correspond two different elements $y_1 = f(x_1)$ and $y_2 = f(x_2)$ of R_f, then the function can be inverted. We can then write

$$x = f^{-1}(y),$$

where f^{-1} is a function from R_f onto the set A.

In the latter case the function f (and likewise its *inverse* f^{-1}) introduces a one-to-one correspondence between the elements x of A and y of R_f. This can be indicated by writing

$$x \longleftrightarrow y \quad \text{is} \quad 1\text{—}1.$$

Obviously, a function f may have an inverse f^{-1} only if the domain D_f and the range R_f have the same cardinality.

If f is a function from A onto B, and g a function from B onto C, we can form a *composite* function gf from A onto C. It is defined by

$$(gf)(x) = g(f(x)).$$

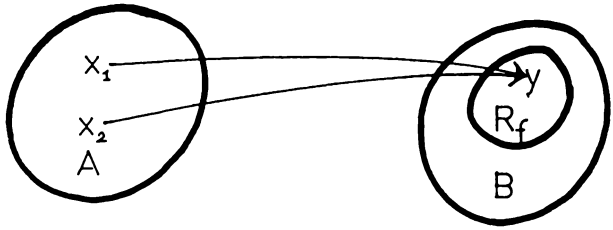


Fig. 1. A function having no inverse

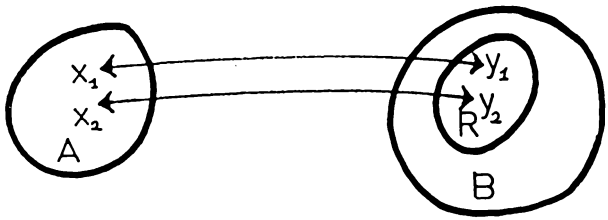


Fig. 2. A function having an inverse

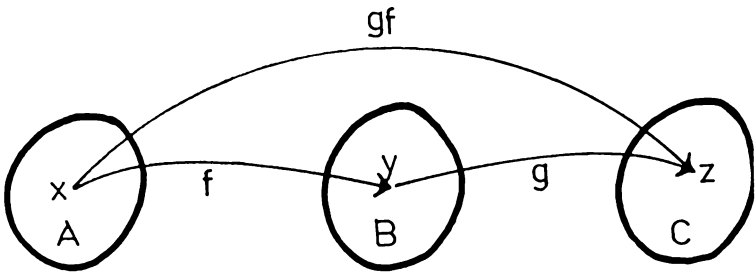


Fig. 3. A composite function

If $C = A$ and $g = f^{-1}$ the composite $f^{-1}f$ is a function from A onto the set A itself. For the value of this function corresponding to the argument x we get: $(f^{-1}f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$. We can also form

another composite function f^{-1} for which we get $(ff^{-1})(y) = f(f^{-1}(y)) = f(x) = y$. These results are denoted in short by writing

$$f^{-1}f = ff^{-1} = 1,$$

where 1 is used as a symbol for the identity function. Such a function is defined by

$$1(x) = x \text{ for all } x \in D_1,$$

D_1 being the domain of definition of the function 1.

3 / Function Space

Just as when dealing with a given set A we are usually interested also in the set $F(A)$ of all subsets of A, when dealing with functions from a set A to a set B we are often interested also in the set $F(A,B)$ of all functions from A to B. This is in particular the case when B is the set of all real numbers. We have then the set $F(A,R)$ of all functions from a given arbitrary set A to the set R of all real numbers:

$$F(A,R) = \{f;\ f: A \to R\}. \tag{1}$$

If f_1 and f_2 are two functions from A to R, and if $f_1(x)$ and $f_2(x)$ are the two real numbers which are the values of these two functions for one and the same argument x, we can add the two real numbers together, and define a function $f_1 + f_2$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{for all } x \in A.$$

The function $f_1 + f_2$ so defined is also a function from A to R.

In a similar way, using the fact that real numbers can also be multiplied by one another and not only added to one another, we define for every function f from A to R and an arbitrary but fixed real number k a function kf by

$$(kf)(x) = kf(x) \quad \text{for all } x \in A.$$

The function kf is then also a function from A to R.

Since the functions $f_1 + f_2$ and kf are functions from A to R, they belong to the set $F(A,R)$ of all functions from A to R. Thus we observe that by using the above rules we can add together two arbitrary functions belonging to $F(A,R)$, and get a function which belongs to this same set $F(A,R)$. And we can multiply any arbitrary function belonging to $F(A,R)$ by an arbitrary real number k, and get a function which

belongs to $F(A,R)$. This is expressed by saying that the set $F(A,R)$ of all functions from A to R forms a *function space.*

If $f_1,f_2,\ldots,f_n$ are n functions from the function space $F(A,R)$, the function f defined by

$$f = a_1f_1 + a_2f_2 + \ldots + a_nf_n,$$

where $a_1,a_2,\ldots,a_n$ are real numbers, is the *linear combination* of $f_1,f_2,\ldots,f_n$ with the *weights* $a_1,a_2,\ldots,a_n$. Of course such a linear combination is to be understood as the function whose values are given by[1]

$$f(x) = a_1f_1(x) + a_2f_2(x) + \ldots + a_nf_n(x) \text{ for all } x \in A.$$

All the linear combinations of given n functions $f_1, f_2,\ldots,f_n$ form an n-dimensional *linear space* L_n,

$$L_n = \{f; f = a_1f_1 + a_2f_2 + \ldots + a_nf_n\,;\, a_i \in R\}, \tag{2}$$

provided that no one of the functions $f_1,f_2,\ldots,f_n$ is a linear combination of the other ones. The functions $f_1,f_2,\ldots f_n$ are then said to be *linearly independent* of one another, and form a *basis system* in the subspace L_n.

A function φ from a function space $F(A,R)$ to the set R of real numbers is called a functional defined on $F(A,R)$. It is a *linear functional* if

$$\varphi(af + bg) = a\varphi(f) + b\varphi(g)$$

for any two functions f and g from $F(A,R)$, and any given two real numbers a and b.

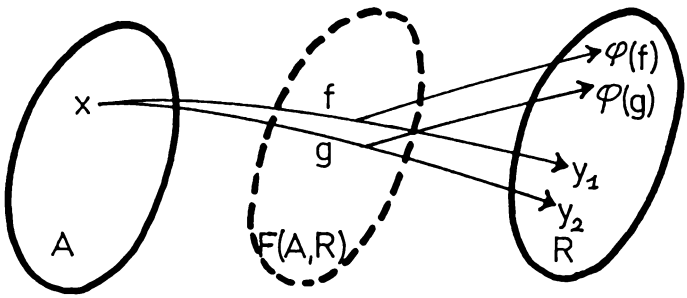


Fig. 4. A functional φ on the function space $F(A,R)$

1. For an infinite n there are other possibilities too, for the association of f with $a_1f_1+\ldots+a_nf_n$, leading to different topologies in function space. We shall pass by the topological problems of an infinite-dimensional function space here.

A function φ from the cartesian product space $F(A,R)\times F(A,R)$ to R is a *bilinear functional* on $F(A,R)$, if it is linear in both of its arguments:

$$\varphi(af+bg,\,h) = a\,\varphi(f,h) + b\,\varphi(g,h), \text{ and}$$
$$\varphi(f,ag+bh) = a\,\varphi(f,g) + b\,\varphi(f,h).$$

Here f, g, and h are any three functions from $F(A,R)$, and a and b are two arbitrary real numbers. A bilinear functional which is symmetric, that is,

$$\varphi(f,g) = \varphi(g,f)$$

for all $f,g \in F(A,R)$, defines an *inner product* in the function space. The inner product of two functions f and g is denoted by $<f,g>$. Accordingly, we can write

$$\varphi(f,g) = <f,g>,$$

if φ is a symmetric bilinear functional.

A inner product defined on $F(A,R)$ is positive-definite, if

$$<f,f> > 0 \text{ for all } f \in F(A,R) \text{ except that } <f,f> = 0 \text{ for } f = 0,$$

negative-definite, if

$$<f,f> < 0 \text{ for all } f \in F(A,R) \text{ except that } <f,f> = 0 \text{ for } f = 0,$$

and indefinite otherwise. A positive-definite, a negative-definite, or an indefinite inner product is said to define in the function space $F(A,R)$ a positive-definite, a negative-definite, or an indefinite *metric*, respectively.

In a positive-definite metric we call the square root of $<f,f>$ the norm of the function f, and denote it by

$$|f| = \sqrt{<f,f>}.$$

The norm of the difference $f-g$ is called the distance between the functions f and g.

The norm so defined is always positive or zero. Writing the norm square of $af+g$, where a is a real number and f and g two functions from $F(A,R)$, we thus always have:

$$|af+g|^2 = <af+g,\, af+g> = a^2<f,f> + <g,g> + 2a<f,g> \geq 0.$$

The discriminant of the polynomial gives at once the inequality of Schwartz,

$$|<f\cdot g>| \leqq |f|\cdot|g| \quad \text{for any } f,g \in F(A,R), \tag{3}$$

which is thus valid in a positive-definite metric.

4 / Vector Space

All that was said above on the function space $F(A,R)$ has simple geometrical interpretation, if the basic set A is finite. If A has a finite number, n, of elements, we can write: $A = \{x_1,x_2,\ldots,x_n\}$. The function space $F(A,R)$ is in this case called an n-dimensional *vector space* V_n:

$$F(A,R) = V_n.$$

Obviously, every function f from A to R is now represented by the combination of the n values $f(x_1), f(x_2), \ldots, f(x_n)$. Thus we can write:

$$f = (f(x_1), f(x_2), \ldots, f(x_n)).$$

Such a function is called an n-component *vector* which has the components $f(x_1)$, $f(x_2), \ldots, f(x_n)$. The elements f of the vector space V_n thus are n-component vectors.

We can easily show that the vector space V_n is an n-dimensional linear space. Indeed, a basis system in V_n is given by the n vectors defined by

$$\begin{cases} f_1 = (1,0,0,\ldots,0), \\ f_2 = (0,1,0,\ldots,0), \\ f_3 = (0,0,1,\ldots,0), \\ \ldots\ldots\ldots \\ f_n = (0,0,0,\ldots,1). \end{cases}$$

Any vector $f \varepsilon V_n$ can be expressed, obviously, as a linear combination

$$f = a_1 f_1 + a_2 f_2 + \ldots + a_n f_n,$$

where the weights are given by

$$a_1 = f(x_1),\ a_2 = f(x_2), \ldots, a_n = f(x_n).$$

Secondly, the vectors $f_1,f_2,\ldots,f_n$ are linearly independent of one another. Thus V_n is an n-dimensional linear space. From the last equations we see, furthermore, that an n-component vector $f \varepsilon V_n$ is represented by the combination of the weitghts:

$$f = (a_1,a_2,\ldots,a_n).$$

The addition of functions becomes in a vector space V_n an addition of vectors. If $f = (a_1,a_2,\ldots,a_n)$ and $g = (b_1,b_2,\ldots,b_n)$ are two vectors of V_n, their sum vector is given by

$$f+g = (a_1 + b_1,\ a_2 + b_2, \ldots, a_n + b_n).$$

The scalar multiplication of a function becomes in a vector space V_n a scalar multiplication of a vector:

$$kf = (ka_1,\ ka_2, \ldots, ka_n)$$

defines the vector kf for any $k \in R$ and $f \in V_n$.

The weights $a_1, a_2, \ldots, a_n$ of the vectors $f \in V_n$ in the basis system $f_1, f_2, \ldots, f_n$ are all linear functionals on V_n. This is seen by writing

$$f = a_1(f)f_1 + a_2(f)f_2 + \ldots + a_n(f)f_n,$$

and using the rules of vector addition and scalar multiplication which give:

$$a_i(f+g) = a_i(f) + a_i(g) \quad \text{for any } f,g \in V_n, \quad \text{and}$$
$$a_i(kf) = ka_i(f) \quad \text{for any } f \in V_n;\ i = 1,2,\ldots,n.$$

We can define a bilinear functional φ on V_n by writing

$$\varphi(f,g) = a_1b_1 + a_2b_2 + \ldots + a_nb_n,$$

where $f = a_1f_1 + a_2f_2 + \ldots + a_nf_n$ and $g = b_1f_1 + b_2f_2 + \ldots + b_nf_n$.

This is symmetric, since

$$\varphi(f,g) = a_1b_1 + a_2b_2 + \ldots + a_nb_n = b_1a_1 + b_2a_2 \ldots +$$
$$+ b_na_n = \varphi(g,f).$$

It is also positive-definite, since

$$\varphi(f,f) = a_1^2 + a_2^2 + \ldots + a_n^2 > 0$$
except for the case $a_1 = a_2 = \ldots = a_n = 0$.

Accordingly, this bilinear functional defines a positive-definite inner product in the vector space V_n:

$$\langle f,g \rangle = a_1b_1 + a_2b_2 + \ldots + a_nb_n.$$

In this positive-definite metric the norm of a vector f becomes

$$|f| = \sqrt{\langle f,g \rangle} = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2},$$

and the distance between two vectors f and g becomes

$$|f-g| = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + \ldots + (a_n-b_n)^2}.$$

This positive-definite metric in the vector space V_n is called the *Euclidean metric*. The vector space V_n where such a metric is defined is called an n-dimensional *Euclidean space* E_n.

An n-dimensional Euclidean space is simply a generalization of the 3-dimensional space geometry studied in school mathematics. The 3-dimensional Euclidean space E_3 represents the ordinary geometry of our 3-dimensional physical space. The components (a_1,a_2,a_3) of a 3-component vector f are, in geometrical illustration, the coordinates of the 'point' f. The norm $|f|$ is the ordinary "Euclidean distance" of the point f from the origin of the coordinate system. The formula

$$|f|^2 = a_1^2 + a_2^2 + a_3^2$$

is simply the Pythagorean theorem when applied twice, first to get the length of the projection b of the vector f on the place of the coordinates (a_1,a_2) (see Fig. 5), and then to get the length of the vector f from the triangle on the plane of b and a_3.

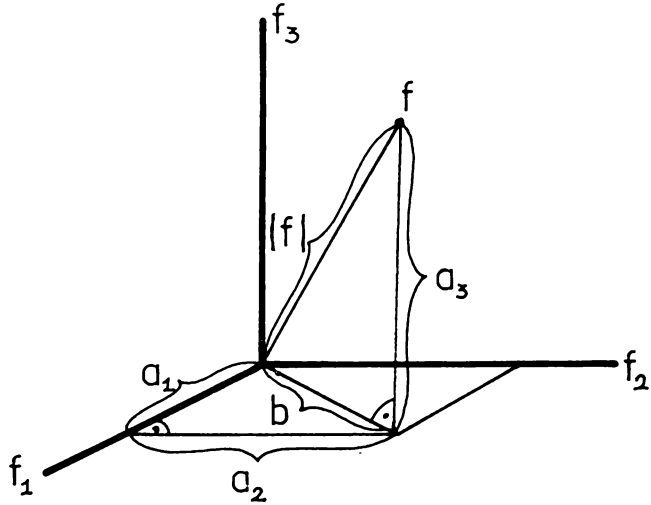


Fig. 5. The norm and the Pythagorean theorem: $|f|^2 = b^2 + a_3^2$ and $b^2 = a_1^2 + a_2^2$.

In the geometrical illustration of the Euclidean space E_3 each vector $f = a_1f_1 + a_2f_2 + a_3f_3 \in E_3$ is represented by the point having the coordinates (a_1,a_2,a_3), or by the arrow pointing to this point and starting from the origin. Let us study the geometrical illustration of the *Euclidean inner product*, by using two such vectors $f = a_1f_1 + a_2f_2 + a_3f_3$ and $g = b_1f_1 + b_2f_2 + b_3f_3$.

Let the angle between vectors f and g (between the arrows representing these two vectors) be ϑ. Then the projection of the vector f along the vector g has the length $|f| \cos \vartheta$, as is evident from the triangle on the plane of f and g, shown in Fig. 6. On the other hand, by the projection theorem we learned at school, we get the same projection of f on g also by summing up the projections of a_1f_1, a_2f_2, and a_3f_3 on the vector g. If α_1 is the angle between g and f_1, the projection of a_1f_1 along g has the length $a_1\cos\alpha_1$. Denoting the angles between g and f_2, and g and f_3, respectively, by α_2 and α_3, we get by projection theorem the equation

$$|f| \cos \vartheta = a_1\cos\alpha_1 + a_2\cos\alpha_2 + a_3\cos\alpha_3.$$

Multiplying by $|g|$ we get

$$|f| \cdot |g| \cos \vartheta = a_1|g| \cos \alpha_1 + a_2|g| \cos \alpha_2 + a_3|g| \cos \alpha_3.$$

But $|g| \cos \alpha_1$ is the length of the projection of g along f_1 so that $|g| \cos \alpha_1 = b_1$. In a similar way we get $|g| \cdot \cos \alpha_2 = b_2$ and $|g| \cos \alpha_3 = b_3$. Thus:

$$|f| \cdot |g| \cos \vartheta = a_1b_1 + a_2b_2 + a_3b_3 = <f,g>. \tag{4}$$

This formula gives the geometrical interpretation of the Euclidean inner product in an Euclidean 3-dimensional space E_3. According to this interpretation the Euclidean inner product of two vectors f and g is the product of the lengths of the vectors f and g times the cosine of the angle ϑ between these vectors.

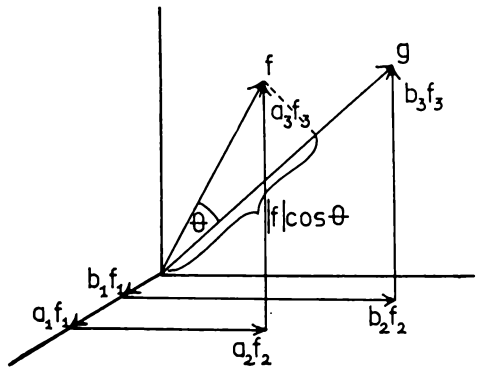


Fig. 6. Geometrical illustration of the Euclidean inner product by means of the projection theorem: $<f,g> = |f| \cdot |g| \cos \vartheta = a_1b_1 + a_2b_2 + a_3b_3$.

We can generalize the last formula to an n-dimensional Euclidean space E_n by writing

$$<f,g> = |f| \cdot |g| \cos \vartheta \quad \text{for any } f,\ g \in E_n,$$

thus defining the angle ϑ between two vectors f and g in an n-dimensional Euclidean space E_n. The condition of orthogonality of two vectors f and g, $\cos \vartheta = 0$, then preserves its meaning even in an n-dimensional space:

$$<f,g> = 0 \quad \text{or} \quad \cos \vartheta = 0$$

means the orthogonality of f and g.

5 / Mathematical Relation

Observation of the specific proportions in which the elements of some sets are with one another does not always mean, even though it often does, the observation of some functional relationships. A more general expression for such a specific proportion is mathematical relation.

Let us first introduce mathematical relation as a generalization of function. If f is a function from a set A to a set B, this means that we are given a set $\{(x, y)\} \subset A \times B$ of pairs (x, y) where x represents an argument and y the corresponding value of the function f. In each pair the first member, the argument x, determines uniquely the second member, the value y. In fact, a function f does not mean anything but a particular rule for picking out the accepted pairs (x, y) from all the elements of $A \times B$. Thus we can write

$$f = f\{(x,y)\} \subset A \times B$$

indicating so that the function f is particular rule for picking out the elements (x,y) of the set $f\{(x,y)\}$.

In the case of a function the set $\{(x, y)\}$ of the elements picked out from $A \times B$ obeys the further condition that in each pair the first member x determines uniquely the second member y. If we give up this condition we have a general case of a mathematical relation defined in the set $A \times B$. Accordingly a *mathematical relation* defined in $A \times B$ is a subset $R\{(x,y)\}$ of the elements (x,y) of $A \times B$ which are picked out by means of a particular rule R from the set $A \times B$:

$$R\{(x,y)\} \subset A \times B.$$

The word "relation" sometimes refers to the set $R\{(x,y)\}$ itself, and sometimes to the rule R used for picking out the elements of $R\{(x,y)\}$.

A mathematical relation of the type just discussed is a two-member relation. We can at once generalize to the case where there are n members in a relation. We define an n-member mathematical relation in a set $A_1 \times .. \times A_n$ as a subset composed of some elements $(x_1, . . ,x_n)$ of $A_1 \times .. \times A_n$:

(5) $\qquad R\{(x_1, . . ,x_n)\} \subset A_1 \times . . \times A_n.$

This definition can be further generalized by considering, instead of a sequence of the elements $x_i \in A_i$ some subsets $X_i \subset A_i$. Then we get an n-number mathematical relation defined in the set $F(A_1) \times . . \times F(A_n)$:

$$R\{(X_1, . . ,X_n)\} \subset F(A_1) \times . . \times F(A_n).$$

In a general case, just as in the case of two-member relation, we may refer by the word "relation" either to the set $R\{\}$ or to the rule R for picking out the elements of the set $R\{\}$.

6 / Invariance and Transformation

Obviously a mathematical relation is a very general notion in terms of which we can express a great variety of specific proportions that we observe in reality. Once we have observed such proprotions, and constructed the corresponding mathematical relation, we are often confronted with the problem of the *invariance* of the observed relation with respect to some *transformations*. The problem of invariance and transformation can be represented mathematically as follows.

Let us consider a function f from a set A onto itself. Such a function means just a permutation of the elements of A with one another, and has always an inverse f^{-1}. Let us call this kind of function a transformation in the set A.

A transformation f in the set A also induces a permutation of the subsets of A with one another. If A_0 is one of these subsets, let us write $f(A_0)$ for the subset which is obtained from A_0 by the transformation f, i.e. $f(A_0) = \{f(x); x \in A_0\}$.

If we have a mathematical relation $R\{(x_1, . . ,x_n)\}$ defined in the set $A \times .. \times A = A^n$, we call this relation invariant with respect to the transformation f, if

$$R\{(x_1, . . ,x_n)\} = \{(f(x_1), . . ,f(x_n)); (x_1, . . ,x_n) \in R\}.$$

In other words, a relation R is invariant with respect to f, if the function f merely permutes the sequences $(x_1, . . ,x_n)$ included in $R\{(x_1, . . ,x_n)\}$ with one another but does not change the set $R\{(x_1, . . ,x_n)\}$ itself.

The invariance of a relation $R\{(X_1, \ldots, X_n)\}$ holding between some subsets of A can be defined in a corresponding way:

$$R\{(X_1, \ldots, X_n)\} = \{(f(X_1), \ldots, f(X_n)); (X_1, \ldots, X_n) \in R\}.$$

If we write $y_i = f(x_i)$ and $Y_i = f(X_i)$ we can write the last two equations in a form

$$R\{(x_1, \ldots, x_n)\} = R\{(y_1, \ldots, y_n);\ y_i = f(x_i)\},$$
$$R\{(X_1, \ldots, X_n)\} = R\{(Y_1, \ldots, Y_n);\ Y_i = f(X_i)\}.$$

But since $y = f(x)$ defines a one-to-one correspondence between the y-elements and the x-elements, and since this same correspondence is committed by the inverse function $x = f^{-1}(y)$, we can write the last two equations in another form

$$R\{(y_1, \ldots, y_n)\} = R\{(x_1, \ldots, x_n);\ x_i = f^{-1}(y)\}.$$
$$R\{(Y_1, \ldots, Y_n)\} = R\{(X_1, \ldots, X_n);\ X_i = f^{-1}(Y_i)\}.$$

These relations can be further written as

$$R\{(y_1, \ldots, y_n)\} = \{(f^{-1}(y_1), \ldots, f^{-1}(y_n));\ (y_1, \ldots, y_n) \in R\},$$
$$R\{(Y_1, \ldots, Y_n)\} = \{(f^{-1}(Y_1), \ldots, f^{-1}(Y_n)); (Y_1, \ldots, Y_n) \in R\},$$

which show that the relation R is invariant also with respect to the inverse transformation f^{-1}.

We can proceed a little further by showing that if f and g are two transformations defined in A, and if the relation R is invariant with respect to both f and g, then R is invariant also with respect to the composite transformation fg (and gf). Indeed, if f merely permutes the elements of R with one another, and if g is another permutation of these same elements, then gf (and fg) is just a permutation composed of the successive permutations f and g, and thus leaves certainly the set R invariant. This can be easily checked by formal calculation in a similar manner as we proved above the invariance of R with respect to f^{-1}.

If we have a set $G = \{f\}$ of transformations defined in A, such that

1° the identity transformation 1 belongs to G,

2° for each transformation $f \in G$ its inverse f^{-1} also belongs to G, and

3° for any two transformations $f \in G$ and $g \in G$ also their composite transformation fg (and gf) belongs to G,

the set G is called a *group of transformations*.

We have above shown that a mathematical relation defined for the elements (or subsets) of A, and invariant with respect to some transformations f in A, is always invariant with respect to a certain group G of transformations. In fact the notion of invariance and the notion of a group of transformations are equivalent. Once a group of transformations G is given we can identify this group by giving mathematical relations which are invariant in the transformations of the group. These relations are the invariants of the group G. Vice versa, if some mathematical relations are given, we can ask for the group of transformations G in which these relations are invariant. This group sometimes reduces to a set containing only the identity transformation 1, which is of course a trivial case from the point of view of invariance. A non-trivial case is in question when the group G, called the invariance group of the relations in question, does not reduce to mere identity transformation.

7 / Algebraic Operation

We shall now study what is meant by an algebraic operation. An algebraic operation is a particular kind of function, viz. a function α from a set $A \times A$ to the set A itself:

$$\alpha : A \times A \to A,$$
$$(x,y) \xrightarrow{\alpha} z,$$

where x,y, and z are elements of the given set A.

Accordingly, when we have a given set A, we can define an algebraic operation α by associating with each pair (x,y) of the elements of A one and only one element $z = \alpha(x,y)$ of the same set A.

An algebraic operation α is called *associative* if it can be continued to a function α^* from $A \times A \times A$ to A in a uniquely determined way, i.e. if

$$\alpha^*(x,y,z) = \alpha(\alpha(x,y),z) = \alpha(x,\alpha(y,z)).$$

We observe that an associative algebraic operation associates with any sequence $(x,y,z,u,\ldots)$ of elements of A one and only one element of A. This element depends only on the elements $x,y,z,u,\ldots$ and of their mutual order in the sequence.

An algebraic operation α is called *commutative* if its value is indeppendent of the order of the arguments, i.e. if

$$\alpha(x,y) = \alpha(y,x)$$

for any given elements x and y of A.

If we have two algebraic operations α and β defined for the same set A, and if at least α is commutative, we say that β is *distributive* with respect to α, if

$$\beta(x,\alpha(y,z)) = \alpha(\beta(x,y),\beta(x,z))$$

for any elements x,y, and z of A.

Let an associative and commutative algebraic operation α be defined in a set A. If there is an element x_α^0 of A, such that

$$\alpha(x_\alpha^0,x) = x$$

for any element x of A, the element x_α^0 is called the *zero element* of the operation α. If, for each element x of A, there is an element x_α^{-1} such that

$$\alpha(x,x_\alpha^{-1}) = x_\alpha^0,$$

the element x_α^{-1} is the *inverse* of x with respect to the operation α.

A set A, in which an associative and commutative algebraic operation α is defined so that the zero element x_α^0 and the inverses x_α^{-1} are uniquely determined, is called an *Abelian group*. An example of such a group is the set of all integers where addition satisfies these requirements. Indeed, the addition associates with any pair (n,m) of integers another integer, viz. the sum $n+m$. The addition is associative, since $(n+m)+k = n+(m+k) = n+m+k$, and commutative since $n+m = m+n$. The zero element of addition is 0, and the inverse of an integer n with respect to the addition is $-n$.

Another example of an Abelian group is the set of all rational numbers with respect to multiplication. Indeed, multiplication associates with any pair (r_1,r_2) of rational numbers another rational number, viz. the product r_1r_2. The multiplication of rational numbers is associative, since $(r_1r_2)r_3 = r_1(r_2r_3) = r_1r_2r_3$, and it is also commutative since $r_1r_2 = r_2r_1$. The zero element of multiplication is 1, and the inverse of a rational number r with respect to multipilication is the rational number $\frac{1}{r}$.

A set A in which an associative and commutative algebraic operation α and an associative algebraic operation β, which is distributive with respect to α, are defined, is called a *ring* provided that the zero element x_α^0 and the inverses x_α^{-1} are uniquely determined. The operation α is called ring addition, and the operation β ring multiplication. A ring may or may not have a zero element x_β^0 or the inverses x_β^{-1} with respect to β-operation. The β-operation may or may not be commutative. In the former case the ring is called commutative. An example of a com-

mutative ring is the set of all integers, when α is defined by addition and β by multiplication. There is a zero element x_α^0, viz. 0, and a zero element x_β^0, viz. 1, in this ring but there are in general no inverses $x\beta^{-1}$, since the inverse $\frac{1}{n}$ of an integer n with respect to multiplication is not an integer itself (except when $n = 1$).

A ring where even the element x_β^0 and the inverses $x\beta^{-1}$ exist and are uniquely determined is called a *field of numbers*. In a field of numbers the element x_α^0 is called zero, and the element x_β^0 is called the unit. In a field of numbers we can speak of four fundamental operations, which are called and denoted as follows:

$$\text{addition: } \alpha(x,y) = x+y,$$
$$\text{subtraction: } \alpha(x,y_\alpha^{-1}) = x-y,$$
$$\text{multiplication: } \beta(x,y) = xy, \text{ and}$$
$$\text{division: } \beta(x,y\beta^{-1}) = \frac{x}{y}\cdot$$

An example of a field of numbers is the field of rational numbers, or the field of real numbers.

In the preceding sections we have already met an algebraic notion which we shall now define in more general terms.

A set A is called a *linear space* over the field K of numbers, if 1^0 A is an Abelian group with respect to an operation α, and if 2^0 a commutative algebraic operation γ from $K\times A$ to A is defined, γ being distributive with α:

$$\gamma\colon K\times A \to A,$$
$$(k,x) \xrightarrow{\gamma} y = \gamma(k,x),$$
$$\gamma(k,x) = \gamma(x,k),$$
$$\gamma(k,\alpha(x,y)) = \alpha(\gamma(k,x), \gamma(k,y)).$$

The operation γ is called *scalar multiplication*, and denoted by $\gamma(k,x) = kx$. Accordingly, we have in a linear space two algebraic operations, viz. addition and scalar multiplication. An example of linear space over the field of real numbers is the function space $F(A,R)$ studied in earlier sections.

A set A is called *general algebra* if it is simultaneously 1^0 a linear space over a field of numbers, and 2^0 a ring, and if 3^0 the ring multiplication is commutative with the scalar multiplication. A trivial example of general algebra is the set of all real numbers. We shall soon meet a less trivial example, viz. matrix algebra.

If a number of algebraic operations, $\alpha_1, \ldots, \alpha_n$, is defined in both the sets A and B, the set B is said to be *homomorphic* to the set A if there is a function f from A to B so that the algebraic operations are invariant: $f(\alpha_i(x,y)) = \alpha_i(f(x),f(y))$ for every operation α_i. If such a function f is one-to-one, the sets A and B are said to be *isomorphic* to one another. Thus isomorphism and homomorphism are algebraic notions (while the 'homeomorphism' to be studied in 5§ is a topological notion).

3 § *Matrices*

1 / Matrix Algebra

A matrix is a table of numbers containing, say, n rows and m columns. If the number belonging to the j^{th} row and the k^{th} column is denoted by a_{jk}. we can write such a matrix as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots\ldots & a_{1m} \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots\ldots & a_{2m} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots\ldots\ldots\ldots & a_{nm} \end{pmatrix} = \text{an } n \times m \text{ matrix.}$$

As indicated above a matrix containing n rows and m columns is referred to as an "$n \times m$ matrix". The numbers a_{jk}, or the elements of the matrix, may be taken from any fixed field of numbers.

We shall restrict ourselves first to square matrices, i.e. to matrices where are as many rows as there are columns: $n = m$. We shall also restrict ourselves to real matrices, i.e. to matrices whose elements are real numbers.

Let $M(n \times n)$ be the set of all real $n \times n$ matrices. We shall show that this set forms a general algebra with respect to certain algebraic operations α, β, and γ. To show this we must show 1) that $M(n \times n)$ is a linear space over the field R of real numbers with respect to an addition α and a scalar multiplication γ, 2) that $M(n \times n)$ can be extended to a ring by defining a ring multiplication β, and 3) that β and γ are commutative with one another.

For the first task we must show a) that $M(n \times n)$ is an Abelian group with respect to an addition operation α, and b) that a scalar multiplication γ can be defined as a function from $R \times M(n \times n)$ to $M(n \times n)$.

For this purpose we define an addition $\alpha(A,B) = A+B$ of any two matrices $A \varepsilon M(n\times n)$ and $B \varepsilon M(n\times n)$ by

$$\underbrace{\begin{pmatrix} a_{11}\ a_{12}\ .\ .\ a_{1n} \\ a_{21}\ a_{22}\ .\ .\ a_{2n} \\ \cdots\cdots\cdots \\ a_{n1}\ a_{n2}\ a_{nn} \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} b_{11}\ b_{12}.\ .\ b_{1n} \\ b_{21}\ b_{22}.\ .\ b_{2n} \\ \cdots\cdots\cdots \\ b_{n1}\ b_{n2}\ .\ .\ b_{nn} \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12}\ .\ .\ a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22}\ .\ .\ a_{2n}+b_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2}\ .\ .\ a_{nn}+b_{nn} \end{pmatrix}}_{A+B}$$

In other words, the sum $A+B$ of the matrices A and B is obtained by adding together the corresponding elements of the matrices A and B. If the element in the j^{th} row and the k^{th} column in the matrix $A+B$ is denoted c_{jk}, we thus has the following rule:

$$c_{jk} = a_{jk}+b_{jk}.$$

Since

$$(a_{jk}+b_{jk})+c_{jk} = a_{jk}+(b_{jk}+c_{jk}) = a_{jk}+b_{jk}+c_{jk},$$

and

$$a_{jk}+b_{jk} = b_{jk}+a_{jk},$$

the addition of matrices so defined is associative and commutative. Thus we can write $(A+B)+C = A+(B+C) = A+B+C$ and $A+B = B+A$. The zero matrix 0 can be defined as a matrix all whose elements are zero. The inverse of a matrix A with respect to addition can be defined as a matrix $-A$ composed of the elements $-a_{jk}$, the a_{jk} being the elements of the matrix A. The set $M(n\times n)$ of square matrices is then an Abelian group with respect to the addition.

A scalar multiplication of any matrix A of $M(n\times n)$ by a real number r can be defined by

$$r \cdot \underbrace{\begin{pmatrix} a_{11}\ a_{12}\ .\ .\ a_{1n} \\ a_{21}\ a_{22}\ .\ .\ a_{2n} \\ \cdots\cdots\cdots \\ a_{n1}\ a_{n2}\ .\ .\ a_{nn} \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} ra_{11}\ ra_{12}\ .\ .\ ra_{1n} \\ ra_{21}\ ra_{22}\ .\ .\ ra_{2n} \\ \cdots\cdots\cdots\cdots \\ ra_{n1}\ ra_{n2}\ .\ .\ ra_{nn} \end{pmatrix}}_{rA}$$

Since $ra_{jk} = a_{jk}r$ this operation is commutative, and since $r(a_{jk}+b_{jk}) = ra_{jk}+rb_{jk}$ it is also distributive with matrix addition. Thus we can write $rA = Ar$ and $r(A+B) = rA+rB$. Accordingly, the set $M(n\times n)$ is a linear space over the field of real numbers.

The second task is to construct a ring multiplication β such that $M(n\times n)$ becomes a ring with respect to the addition α and the ring

multiplication β. For this purpose we define a matrix multiplication $\beta(A,B) = AB$ of any two matrices $A \varepsilon M(n \times n)$ and $B \varepsilon M(n \times n)$ by the following rule:

$$c_{jk} = \sum_{i=1}^{n} a_{ji} b_{ik},$$

where c_{jk} is the element of j^{th} row and the k^{th} column in the matrix $C = AB$, and a_{jk} and b_{jk} are the corresponding elements in the matrices A and B, respectively. In words: we get the element c_{jk} of the matrix AB by computing the "product sum" between the j^{th} row of A and the k^{th} column of B, as indicated below:

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}} = \underset{AB}{\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}}.$$

For instance, if we have two 3×3 matrices

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

then the element c_{11} of $C = AB$ is obtained from the elements of first row of A and first column of B in the following way:

$$c_{11} = 1 \cdot 2 + 3 \cdot 3 + 2 \cdot 1 = 13.$$

The element c_{12} is obtained from the first row of A and second column of B in a similar way:

$$c_{12} = 1 \cdot 1 + 3 \cdot 1 + 2 \cdot 2 = 8.$$

Since

$$\sum_k \left(\sum_l a_{jl} b_{lk} \right) c_{kl} = \sum_l a_{jl} \left(\sum_k b_{lk} c_{kl} \right) = \sum_l \sum_k a_{jl} b_{lk} c_{kl},$$

the matrix multiplication is associative. Thus we can write $(AB)C = A(BC) = ABC$ for any three matrices A, B, and C from $M(n \times n)$. It follows from this that we have a definite matrix $A_1 A_2 \dots A_n$ as the product of any n matrices $A_1, \dots, A_n$ from $M(n \times n)$.

However, since the "product sum" of the j^{th} row of A and the k^{th} column of B is not necessarily the same as the "product sum" of the j^{th} row of B and the k^{th} column of A, the matrix multiplication is not commutative. Thus in general AB and BA are different matrices.

But commutativity is not a necessary condition for ring multiplication.

Since

$$\sum_i a_{ji}(b_{ik}+c_{ik}) = \sum_i a_{ji}b_{ik} + \sum_i a_{ji}c_{ik},$$

the matrix multiplication is distributive with respect to matrix addition. Thus we can write $A(B+C) = AB+AC$.

It follows from the last results that $M(n\times n)$ indeed is a ring with respect to the matrix multiplication and the matrix addition defined above. It remains to be shown that matrix multiplication is commutative with scalar multiplication of matrices. We have to show that $rAB = = ArB = ABr$ for any two matrices A and B from $M(n\times n)$ and for any real number r. But this is evident, since

$$r\sum_i a_{ji}b_{ik} = \sum_i a_{ji}rb_{ik} = \sum_i a_{ji}b_{ik}r.$$

Thus we have completed the proof that $M(n\times n)$ is a general algebra with respect to the operations of matrix addition, matrix multiplication, and scalar multiplication of matrices defined above.

If we relax the conditions of general algebra we can define algebraic operations for non-square matrices. For instance, we observe that the set $M(n\times m)$ of all $n\times m$ real matrices is a linear space over the field of real numbers. Indeed, we can at once generalize the addition defined above for square matrices to this non-square case. Thus we can add up any two $n\times m$ matrices $A(n\times m)$ and $B(n\times m)$. We can also extend the scalar multiplication to these matrices so that any $n\times m$ matrix $A(n\times m)$ can be multiplied by a real number r to get the matrix $rA = Ar$.

We observe also that a product $A(n\times m)B(m\times k)$ can always be computed for an $n\times m$ matrix $A(n\times m)$ and an $m\times k$ matrix $B(m\times k)$, when n,m, and k are arbitrary positive integers.

A matrix is often denoted by indicating its "general element", i.e. the element of, say the j^{th} row and the k^{th} column. The matrix is then written as a collection of these elements: $A = \|a_{jk}\|$.

2 / Linear Transformation

The usefulness of the matrix algebra $M(n\times n)$ is due to the equivalence of $n\times n$ matrices with linear transformations in an n-dimensional vector space V_n.

We have (see p. 29) introduced the vector space V_n as a function space $F(A,R)$, where the basic set A has n elements $x_1,x_2,\ldots,x_n$, n being a finite number. Each function f from A to R is a combination of the

n values $f(x_1),f(x_2),\ldots,f(x_n)$. Denoting these values by $f(x_1)=a_1$, $f(x_2)=a_2,\ldots,f(x_n)=a_n$ we could express each function f as a combination $f=(a_1,\ldots,a_n)$ of the n real numbers $a_1,a_2,\ldots,a_n$. We shall now write the components of this n-component vector f as a column, and denote this column by x. So we have the one-to-one correspondence between all the vectors $f \in V_n$ and the $n\times 1$ matrices $x \in M(n\times 1)$:

$$f=(a_1,a_2,\ldots,a_n) \longleftrightarrow x=\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \text{ (one-to-one).}$$

We can even define the addition of vectors, and the scalar multiplication of a vector, by the respective matrix operations:

$$x+y \quad \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \cdot \\ \cdot \\ a_n+b_n \end{bmatrix} \quad \text{and} \quad kx=\begin{bmatrix} ka_1 \\ ka_2 \\ \\ \cdot \\ ka_n \end{bmatrix},$$

where x and y are the columns composed of the numbers $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$, respectively. Then we can represent the base vectors $f_1,f_2,\ldots,f_n$ by the respective columns $e_1,e_2,\ldots,e_n$ as follows

$$f_1 \longleftrightarrow e_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad f_2 \longleftrightarrow e_2=\begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \ldots, \quad f_n \longleftrightarrow e_n=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

Now the representation of the vector f as a linear combination $f=a_1f_1+\ldots+a_nf_n$ becomes expressed by

$$x=a_1e_1+a_2e_2+\ldots+a_ne_n.$$

Accordingly, all the vector operations in a vector space V_n can be translated to matrix operations with the $n\times 1$ matrices of the set $M(n\times 1)$ in a unique way. The latter set of matrices is thus a one-to-one matrix

representation of the vector space V_n. We shall call it the representation of V_n by "*column vectors*", and denote it by W_n:

$$M(n\times 1) = W_n \longleftrightarrow V_n \text{ (one-to-one).}$$

Another matrix representation of the vector space V_n is given by the matrix set $M(1\times n)$. From every column vector x we get an element of $M(1\times n)$, or a "*row vector*", just by writing the components of x along a row:

$$x \longleftrightarrow x' = (a_1\ a_2 \ldots\ldots a_n).$$

Even this correspondence, of course, is one-to-one. Matrix addition and the multiplication of a matrix by a scalar again give the corresponding vector operations. The base vectors are now given by

$$\begin{aligned} e_1' &= (1\ \ 0\ \ 0\ \ \ldots\ \ 0),\\ e_2' &= (0\ \ 1\ \ 0\ \ \ldots\ \ 0),\\ &\ldots\ldots\ldots\ldots\ldots\\ e_n' &= (0\ \ 0\ \ 0\ \ \ldots\ \ 1). \end{aligned}$$

The representation of the vector space V_n by the row vectors $x' \in M(1\times n)$ is called *dual* to the representation W_n by the column vectors. The vector space spanned by the row vectors is called dual to the space W_n, and denoted by W_n':

$$M(1\times n) = W_n' \longleftrightarrow V_n \text{ (one-to-one).}$$

Let us now consider the matrix algebra $M(n\times n)$. Let A be one of its elements, that is, a matrix composed of n rows and n columns of real numbers. (Of course, this A should not be confused with the basic set of the function space $F(A,R)$ which was denoted by the same letter.) The matrix product of such an $n\times n$ matrix A and an n-component column vector x is again an n-component column vector. Let us denote it by y:

$$y = \begin{bmatrix} b_1\\ b_2\\ \cdot\\ \cdot\\ b_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11}\ a_{12}\ \ldots\ a_{1n}\\ a_{21}\ a_{22}\ \ldots\ a_{2n}\\ \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\\ a_{n1}\ a_{n2}\ \ldots\ a_{nn} \end{bmatrix}}_{A} , \underbrace{\begin{bmatrix} a_1\\ a_2\\ \cdot\\ \cdot\\ a_n \end{bmatrix}}_{x} .$$

When written explicitly for the components of y this gives, according to the rules of matrix multiplication:

$$\begin{aligned} b_1 &= a_{11}a_1+a_{12}a_2+\ldots+a_{1n}a_n,\\ b_2 &= a_{21}a_1+a_{22}a_2+\ldots+a_{2n}a_n,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ b_n &= a_{n1}a_1+a_{n2}a_2+\ldots a_{nn}a_n. \end{aligned}$$

When every vector $x \in W_n$ is multiplied by the same $n\times n$ matrix A in this way, we get all these vectors transformed to the corresponding y-vectors within the space W_n. Thus such an $n\times n$ matrix A determines a transformation of the vector space W_n to itself. We call it a *linear transformation* of this space.

The above equation written explicitly for the $b_1,b_2,\ldots,b_n$ can be expressed in matrix form either as $y = Ax$ or as $y' = x'A'$. Here A' is the 'transpose' of A, obtained from the matrix A by writing the rows as columns and vice versa. Thus to every linear transformation $x \longrightarrow y = Ax$ in the space W_n there corresponds the linear transformation $x' \longrightarrow y' = x'A'$ in the dual space W_n':

$$x \longrightarrow y = Ax \quad \longleftrightarrow \quad x' \longrightarrow y' = x'A'.$$

From the equation $y' = (Ax)' = x'A'$ we see that $(Ax)' = x'A'$. This rule can be generalized to any matrix product AB so that we have in general:

$$(AB)' = B'A'.$$

Let us now study under which conditions a linear transformation $x \longrightarrow y = Ax$, or, what is the same, $x' \longrightarrow y' = x'A'$, is one-to-one. The domain D_A of the transformation A is, of course, the vector space W_n:

$$D_A = W_n = \{x\,;\, x = a_1e_1+\ldots+a_ne_n\,;\, a_i \in R\}.$$

The range of the transformation A, on the other hand, is

$$R_A = \{y\,;\, y = Ax\,;\, x \in W_n\} \subset W_n.$$

We have to study under which conditions $R_A = D_A$.

For this purpose we can express y in terms of the base vectors $e_1,e_2,\ldots,e_n$:

$$y = Ax = A(a_1e_1+\ldots+a_ne_n) = a_1(Ae_1)+\ldots+a_n(Ae_n).$$

Evidently, if the transformed base vectors $Ae_1,Ae_2,\ldots,Ae_n$ are linearly independent of one another, the vectors y, obtained when the weights $a_1,a_2,\ldots,a_n$ run over all real numbers, cover the whole space W_n. In

this case we thus have $R_A = W_n = D_A$, so that the transformation A is one-to-one.

What are the transformed base vectors $Ae_1, Ae_2, \ldots, Ae_n$? Performing the matrix multiplications we find that they are the column vectors formed by the columns of the matrix A:

$$h_1 = Ae_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ a_{n1} \end{bmatrix}, \quad h_2 = Ae_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ a_{n2} \end{bmatrix}, \quad \ldots, h_n = Ae_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ a_{nn} \end{bmatrix}.$$

The linear independence of these vectors of one another or, as we can say, the linear independence of the columns of the matrix A of one another, is thus the necessary and sufficient condition for the transformation determined by the matrix A being one-to-one.

If we consider, instead of the transformation $x \rightarrow y = Ax$ the dual transformation $x' \rightarrow y' = x'A'$ we find out that another condition, equivalent to the previous one, is that the rows of A must be linearly independent of one another. These two conditions are the same, and are expressed together by saying that the *rank of the matrix* A is n. Such a matrix is called *non-singular.*

3 / The Inverse of Matrix

Every non-singular $n \times n$ matrix A has of course an inverse, that is, there is a uniquely determined $n \times n$ matrix A^{-1} so that $AA^{-1} = A^{-1}A = 1$. Here I is the unit matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ & & & & \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

To show how the inverse matrix is calculated we first define, for any $m \times m$ matrix B, a real number called its determinant and denoted by $|B|$. It is called a determinant of the m^{th} order. The definition and the calculation of a determinant of m^{th} order happens by means of a recursive formula which reduces each determinant of m^{th} order to a weighted sum of determinants of $(m-1)^{\text{th}}$ order. This kind of

recursion determines uniquely the value of any determinant, for we make the further convention that the determinant of an 1×1 matrix containing only an element $a \in R$ is the number a itself: $|a| = a$.

The recursion formula is

$$|B| = \sum_{k=1}^{n} (-1)^{j+k} b_{jk} |B_{jk}|, \text{ with fixed j,} \tag{6}$$

where b_{jk} is the element of the j^{th} row and k^{th} column of the matrix B, and $|B_{jk}|$ is the determinant of the $(k-1)\times(k-1)$ matrix obtained from the matrix B when the j^{th} row and k^{th} column are eliminated:

$$B_{jk} = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{j1} & b_{j2} & \cdots & b_{jk} & \cdots & b_{jn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} & \cdots & b_{nn} \end{vmatrix}$$

Since the order of the determinants $|B_{jk}|$ is $m-1$, this formula gives the desired recursion.

Let us study how the recursion works in practice. Let us calculate the determinant of a 3×3 matrix

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix}.$$

We can choose any row whatever to be the j^{th} row which is the basis of the reduction. Let us choose $j = 1$. Then the first application of the recursion formula gives:

$$B = (-1)^{1+1}\cdot 1\cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + (-1)^{1+2}\cdot 2\cdot \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + \\ + (-1)^{1+3}\cdot 1\cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}.$$

Applying the same recursion then for the remaining determinants of the second order we get:

$$\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = (-1)^{1+1}\cdot 1\cdot 1 + (-1)^{1+2}\cdot 3\cdot 1 = 1-3 = -2,$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = (-1)^{1+1}\cdot 2\cdot 1 + (-1)^{1+2}\cdot 3\cdot 3 = 2-9 = -7,$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = (-1)^{1+1}\cdot 2\cdot 1 + (-1)^{1+2}\cdot 1\cdot 3 = 2-3 = -1.$$

Thus for the determinant $|B|$ we get the result

$$|B| = (-1)^{1+1} \cdot 1 \cdot (-2) + (-1)^{1+2} \cdot 2 \cdot (-7) + (-1)^{1+3} \cdot 1 \cdot \cdot (-1) = -2 + 14 - 1 = +11.$$

Useful formulae holding for determinants are

$$|A'| = |A| \quad \text{and} \quad |AB| = |A| \cdot |B|.$$

Now the inverse of a non-singular $n \times n$ matrix A is simply given by

$$A^{-1} = \|(A^{-1})_{jk}\|, \text{ where}$$

$$(A^{-1})_{jk} = \frac{(-1)^{j+k}|A_{kj}|}{|A|}. \tag{7}$$

Notice the reversed order of subscripts in the sub-determinant:

$$A_{kj} \text{ versus } (A^{-1})_{jk}.$$

We see from the formula that in order that A^{-1} exists the determinant $|A|$ must be different from zero. This is indeed a necessary and sufficient condition for the non-singularity of A. Accordingly, we have three mutually equivalent conditions of non-singularity of A: the linear independence of its rows, the linear independence of its columns, and the non-vanishing of its determinant.

4 / Eigen Values

We can now go into a deeper analysis of the metric of a vector space. Let us represent the general vector space V_n by a column vector space W_n. Then, obviously, every bilinear functional φ on V_n is represented in W_n by a matrix function as follows:

$$\varphi(x,y) = x'Cy. \tag{8}$$

Here C is an $n \times n$ real matrix, and x and y are two column vectors from W_n, x' being the row vector formed as the transpose of x. In particular, the symmetry condition of φ becomes

$$C' = C.$$

Such a matrix is called symmetric. Thus any symmetric $n \times n$ matrix can define a metric in an n-dimensional vector space. When a metric is defined by such a matrix C we can call this matrix the *metric matrix* of the vector space in question.

Let us study the inner product defined by

$$< x,y > = x'Cy$$

in the vector space W_n. In a linear transformation determined by a matrix A the vector x is transformed to Ax and the vector y to the vector Ay. Thus the inner product between x and y is transformed as follows:

$$< x,y > = x'Cy \xrightarrow{A} (Ax)'C(Ay) = x'A'CAy.$$

Thus the metric matrix C has been replaced by the matrix $A'CA$:

$$C \xrightarrow{A} A'CA.$$

We apply now, without proof, the theorem according to which "every symmetric real matrix C can be transformed by an orthogonal transformation A to a diagonal matrix D". The transformation determined by the matrix A is orthogonal when

$$AA' = A'A = I \text{ so that } A' = A^{-1}.$$

The diagonality of D means that D is of the form

$$D = \begin{vmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{vmatrix} .$$

Thus, by the above theorem, we can write

$$A'CA = D, \tag{9}$$

where A is an orthogonal transformation,

According to the above theorem we have the result that every metric in a vector space W_n can be expressed, after a suitable orthogonal transformation, by a diagonal metric matrix D:

$$< x,x > = x'Cy \xrightarrow{A} x'Dy.$$

To study the different metrices of a vector space we thus have to study the different diagonal metric matrices D. The diagonal elements of D are called the *eigen values* of all the symmetric matrices C that can be transformed to D by an orthogonal transformation. Thus, to study the different metrices we have to study only the eigen values of the metric matrices.

For a vector x having the components $a_1, a_2, \ldots, a_n$ we get, using the diagonal representation of the metric matrix:

$$(10) \qquad < x,x > = d_1 a_1^2 + d_2 a_2^2 + \ldots + d_n a_n^2.$$

Since the squares $a_1^2, \ldots, a_n^2$ are positive unless they are zero, the whole expression $< x,x >$ is positive for all $x \neq 0$ if, and only if all the eigen values $d_1, d_2, \ldots, d_n$ are positive. Accordingly, the metric defined by a matrix C is positive-definite if, and only if the eigen values of C are all positive. In a similar way we observe that the metric is negative-definite, if all the eigen values of the metric matrix are negative, and indefinite, if there are both positive and negative eigen values, or if some of the eigen values is zero.

As a particular case of positive-definite metric we have the case where $D = I$, or all the eigen values of the metric matrix are 1. This is the case of the Euclidean metric. An Euclidean metric is invariant with respect to orthogonal transformations:

$$< x,y > = x'Iy \xrightarrow{A} x'A'IAy = x'A'Ay = x'y = < x,y >,$$

when $A'A = I$ or when A is orthogonal.

A symmetric matrix C all of whose eigen values are positive is called a *positive-definite matrix*. Such a matrix has some remarkable formal properties. Since its eigen values are all positive we can construct from the eigen values the matrix

$$D^{1/2} = \begin{pmatrix} \sqrt{d_1} & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{d_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \sqrt{d_n} \end{pmatrix}$$

Defining the matrix S by $S = AD^{1/2}$, where A is the orthogonal matrix which transforms C to D, and using the decomposition

$$D = A'CA \quad \text{or} \quad C = ADA' \quad \text{we get:}$$

$$(11) \qquad C = SS', \; S'S = D$$

The equation $SS' = C$ means, when written for the elements c_{jk} of C and the elements s_{jk} of S, that

$$c_{jk} = \sum_l s_{jl} s_{kl}.$$

In words: the elements of a positive-definite matrix C can always be expressed in a product sum form.

From the inequality of Schwartz, $|< x,y >| \leq |x| \cdot |y|$, we get another

property of the elements of a positive-definite matrix C. Applying this inequality to the unit vectors $e_1, e_2, \ldots, e_n$ we get:

$$|< e_j, e_k >|^2 = |e'_j C e_k|^2 = c_{jk}^2 \leq < e_j, e_j > < e_k, e_k > = c_{jj} c_{kk}.$$

This implies that

$$c_{jj} > 0 \text{ and } c_{jk}^2 \leq c_{jj} c_{kk} \text{ for all } j, k = 1, \ldots, n$$

holds good for the elements of a positive-definite matrix C.

Let us return to the general case where C is a symmetric, not necessarily positive-definite matrix. By the theorem mentioned above it can be transformed, by an orthogonal transformation A, to a diagonal matrix D:

$$A'CA = D, \text{ where } A'A = AA' = I.$$

Its eigen values, or the diagonal elements of D, can be given different interpretations in connection with different mathematical problems.

First, from $A'CA = D$ and $AA' = I$ we get $AA'CA = CA = AD$. Writing the n columns of A as the n column vectors $u_1, u_2, \ldots, u_n$,

$$u_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ a_{n1} \end{pmatrix}, \quad u_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \cdot \\ a_{n2} \end{pmatrix}, \quad \ldots, \quad u_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ a_{nn} \end{pmatrix},$$

we get from $CA = AD$ the equations

$$Cu_1 = d_1 u_1, \quad Cu_2 = d_2 u_2, \quad \ldots, \quad Cu_n = d_n u_n.$$

Accordingly, the eigen values $d_1, d_2, \ldots, d_n$ of C are the n solutions of the equation

$$Cu = \lambda u, \quad \text{where } u \in W_n. \tag{12}$$

This is called the *eigen value equation* of the matrix C. To each eigen value d_k there corresponds an *eigen vector* u_k. If two eigen values are equal, $d_j = d_k$, all the elements of the 2-dimensional vector space spanned by u_j and u_k are eigen vectors belonging to this 2-fold 'degenerate' eigen value. For a j-fold degeneration we get of course a j-dimensional space of eigen vectors.

Secondly, from the eigen value equation we get

$$(C - \lambda I)u = 0.$$

If the matrix $C - \lambda I$ is non-singular, that is, if its determinant is not zero, the inverse matrix $(C - \lambda I)^{-1}$ exists. Then we get

$$(C - \lambda I)u = 0 \rightarrow (C - \lambda I)^{-1}(C - \lambda I)u = 0 \rightarrow Iu = 0 \rightarrow u = 0.$$

Accordingly, if $|C-\lambda I| \neq 0$ the eigen value equation has only the trivial solution $u = 0$. To get the non-trivial solutions we thus have to put

$$|C-\lambda I| = 0. \tag{13}$$

This gives the *characteristic equation*

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & \dots & c_{1n} \\ c_{12} & c_{22}-\lambda & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{1n} & c_{2n} & \dots & c_{nn}-\lambda \end{vmatrix} = 0$$

of the symmetric matrix C. Thus we get the eigen values $d_1, d_2, \dots, d_n$ also as the solutions of the characteristic equation for the unknown λ. It is a polynomial of the degree n in λ.

Obviously, we can consider the eigen values of a nonsymmetric matrix C too, defining them by the eigen value equation or by the characteristic equation.

4 § *Real Functions*

We have so far been discussing notions which all, in one way or another, are related with the function space $F(A,R)$, that is, with functions from a basic set A to the set R of real numbers. In this way we came to the notion of vector space and to the linear operations in a vector space for which matrix operations were constructed.

We shall now confine our scope further, and study the particular case of a function space $F(A,R)$, where even the basic set A is either a subset of the set R of real numbers or of a cartesian product $R \times R \times \dots \times R = R^n$ of the set R of real numbers with itself. The elements

$$f \in F(A,R), \text{ where } A \subset R^n,$$

are called the *real functions of n* variables defined in A.

To each sequence $x_1, x_2, \dots, x_n$ of real numbers from the set A such a function f associates one and only one real number y:

$$f(x_1,x_2,\dots,x_n) = y \in R, \quad (x_1,x_2,\dots,x_n) \in A.$$

The real functions can be analyzed in more detail by means of differential operations, the first of which is derivation.

1 / Derivation

Let us begin with the real functions of one variable, that is, with functions f obeying

$$f \in F(A,R), \quad \text{where } A \subset R.$$

Such a function f associates with every real number $x \in A$ one and only one real number y:

$$f(x) = y \in R, \; x \in A.$$

The function f is called *continuous* in the 'point' x, if the difference $f(x+\Delta x)-f(x-\Delta x)$ approaches zero when the real number Δx approaches zero:

$$f(x+\Delta x)-f(x-\Delta x) \to 0 \quad \text{when } \Delta x \to 0.$$

It is continous in an interval $(a,b) = \{x; a < x < b\} \subset A$ of real numbers, if it is continuous in each point $x \in (a,b)$.

Denoting

$$f(x+\Delta x)-f(x) = \Delta f$$

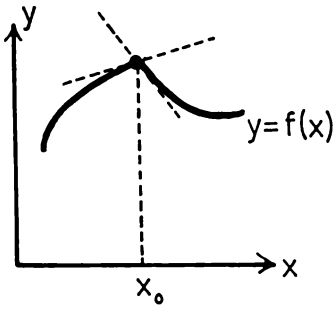
we call a continous function f derivable in the point x, if the ratio $\Delta f/\Delta x$ approaches a uniquely determined real number (positive, negative, or zero, and finite or infinite), when Δx approaches zero. This real number is called the *derivative* of f in the point x, and it is denoted by $\frac{df}{dx}$:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \quad \text{so that} \quad \frac{\Delta f}{\Delta x} \to \frac{df}{dx} \quad \text{when } \Delta x \to 0.$$

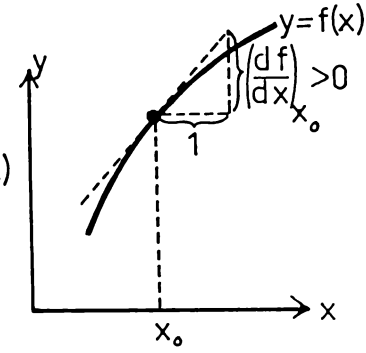
If we want to indicate the point x_0 to which the derivative refers we may write $\left(\frac{df}{dx}\right)_{x_0}$ or $f'(x_0)$:

$$\left(\frac{df}{dx}\right)_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0).$$

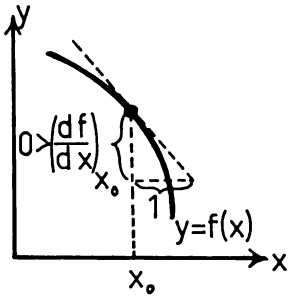
We observe that the derivative of the function f in the point x indicates how steeply this function is increasing or decreasing in the point x (see Fig. 7). If the function f is increasing in the point x_0 its derivative in this point is positive. If the function is decreasing its derivative is negative. And if the function is neither inreasing nor decreasing but has a maximum or a minimum in the point x_0 then its derivative in this point is zero. These cases are illustrated in Fig. 7. The figure also



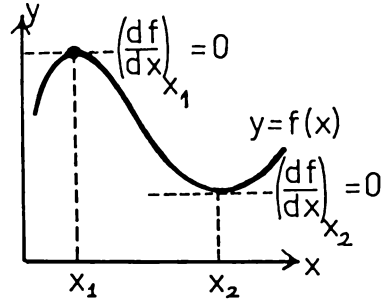
A real function f which is continuous in the point x_0 but has no uniquely determined derivative in this point.



A real function which has a positive derivative in the point x_2.



A real function f which has a negative derivative in the point x_0.



A real function which has a maximum in the point x_1 and a minimum in the point x_2.

Fig. 7.

illustrates how the derivative of the function f in the point x_0 determines the tangent of the curve $y=f(x)$ in the point x_0.

2 / Derivatives of Elementary Functions

It is useful to know the derivatives of the so called elementary functions. For the *power function* $f(x) = x^a$ where a ia a real number, we get at once:

$$\frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^a - x^a}{\Delta x} =$$
$$= \frac{x^a + a \cdot \Delta x \cdot x^{a-1} + <\Delta x^2> - x^a}{\Delta x} \longrightarrow a x^{a-1}$$

when $\Delta x \longrightarrow 0$. Here $<\Delta x^2>$ is a short notation for terms which contain at least the second power of Δx. Accordingly,

$$\frac{dx^a}{dx} = a x^{a-1}.$$

Both the power function and its derivative are defined in the whole set R of real numbers.

For the *exponential function* $f(x) = a^x$, where a is a positive number, we get first:

$$\frac{\Delta f}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x}.$$

We know that $a^0 = 1$. Accordingly, for a small positive Δx the difference $a^{\Delta x} - 1$ is a small positive number (a is always positive). It approaches zero when Δx approaches zero. If we write

$$a^{\Delta x} - 1 = \frac{1}{y},$$

the variable y so defined approaches infinity when Δx approaches zero, and vice versa. Thus by introducing the variable y we can consider the limit process $y \longrightarrow \infty$ instead of the limit process $\Delta x \longrightarrow 0$. It follows from $a^{\Delta x} = 1 + 1/y$ that we have, inversely,

$$\Delta x = \log_a \left(1 + \frac{1}{y}\right).$$

Thus the ratio $\Delta f / \Delta x$ can be expressed as

$$\frac{\Delta f}{\Delta x} = a^x \cdot \frac{1}{y \log_a \left(1 + \frac{1}{y}\right)} = a^x \cdot \frac{1}{\log_a \left(1 + \frac{1}{y}\right)^y}.$$

We have to study what becomes of $(1 + 1/y)^y$ when y approaches infinity.

We shall first study what happens when y approaches infinity through the integer values n:

$$\left(1+\frac{1}{n}\right) \to ? \quad \text{when } n \to \infty.$$

When n is a positive integer we get by means of the binomial expansion known from school:

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2}\cdot\frac{1}{n^2}+$$

$$+\frac{n(n-1)(n-2)}{2\cdot 3}\frac{1}{n^3}+..+\frac{1}{n^n}=2+\frac{1}{2!}\cdot\left(1-\frac{1}{n}\right)+\frac{1}{3!}\cdot$$

$$\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{n-1}{n}\right)\cdot$$

Here we have denoted $1\cdot 2\cdot\ldots\cdot k = k!$. The factors $1-\frac{1}{n}$, $1-\frac{2}{n}$, .. increase when n increases from which we conclude that $\left(1+\frac{1}{n}\right)^n$ also increases when n increases.

On the other hand, $\left(1+\frac{1}{n}\right)^n$ never reaches 3. Indeed, we see this if we observe that every factor $1-\frac{1}{n}$, $1-\frac{2}{n}$, .. approaches 1 when n approaches infinity. Accordingly,

$$\left(1+\frac{1}{n}\right)^n \to 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots \text{ when } n\to\infty.$$

But obviously

$$2+\frac{1}{2!}+\frac{1}{3!}+\ldots<2+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots,$$

since every term of the right side (except the first term) is larger than the corresponding term on the left side. The infinite sum on the right side approaches 3 without reaching it by any finite number of terms. Indeed, we can follow the sum by starting with the number 2, advancing then by adding to it half of the distance (3—2) to get $2\frac{1}{2}$; then add again half of the remaining distance $\left(3-2\frac{1}{2}\right)$ to get $2\frac{1}{2}+\left(\frac{1}{2}\right)^2$, then add half of the remaining distance to get $2\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3$, etc.

Since $\left(1+\frac{1}{n}\right)^n$ increases when n increases but remains always smaller than 3, we conclude that the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

exists, and is a number smaller than 3. This number is called the *Neperian number*. It is one of the most important numbers in differential calculus. We get from the above series at once a series for the computation of e:

$$e = 2+\frac{1}{2!}+\frac{1}{3!}+\ldots \tag{14}$$

By means of this series we can calculate the numerical value of e by any accuracy desired. The result is 2,718281828459... (e is an irrational number).

It remains to be shown that the limit

$$\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = \mathrm{e}$$

holds good even if y is allowed to approach infinity through real numbers in general, i.e. without restricting it to the integer values n. The proof is based on the observation that if n and $n+1$ are the two successive integers between which the value of y is,

$$n \leq y < n+1,$$

then

$$\left(1+\frac{1}{y}\right)^y \leq \left(1+\frac{1}{n}\right)^y < \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n$$

while, on the other hand,

$$\left(1+\frac{1}{y}\right)^y > \left(1+\frac{1}{n+1}\right)^y \geq \left(1+\frac{1}{n+1}\right)^n =$$

$$= \frac{1}{1+\frac{1}{n+1}}\left(1+\frac{1}{n+1}\right)^{n+1}$$

Accordingly, we have

$$\frac{1}{1+\frac{1}{n+1}}\left(1+\frac{1}{n+1}\right)^{n+1} < \left(1+\frac{1}{y}\right)^y < \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n.$$

Since the number $\left(1+\frac{1}{y}\right)^y$ is here between two numbers which both approach e when y (and thus n) approaches infinity, the number $\left(1+\frac{1}{y}\right)^y$ itself also approaches e.

Using the number e we get first for the function $f(x) = a^x$:

$$\frac{\Delta f}{\Delta x} \to a^x \cdot \frac{1}{\log_a e} \text{ when } \Delta x \to 0.$$

However, the number $\log_a e$ means the number x_0 for which the equation $a^{x_0} = e$ holds good. It follows from this equation that

$$a = a^{x_0 \cdot \frac{1}{x_0}} = e^{\frac{1}{x_0}}$$

so that $\frac{1}{x_0} = \log_e a = 1/(\log_a e)$. Accordingly we can write

$$\frac{\Delta f}{\Delta x} \to a^x \log_e a \quad \text{when } \Delta x \to 0.$$

The function $f(x) = \log_e x$ is called the *natural logarithm* function, and the subscript e is then often left off from the notation. Accordingly, we can write our final result in the form

$$\frac{da^x}{dx} = a^x \log a.$$

The exponent function and its derivative are also defined in the whole set R of real numbers.

To calculate the derivatives of the elementary *trigonometric functions* $f(x) = \sin x$ and $f(x) = \cos x$ we can use the geometric construction shown in Fig. 8.

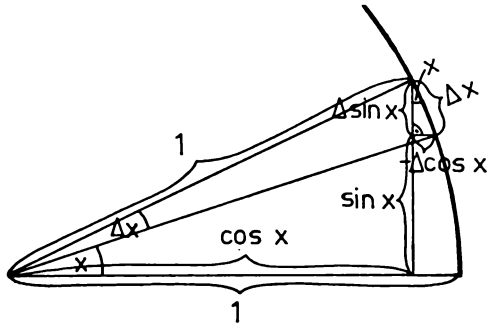


Fig. 8. The differential triangle

This figure shows a "differential rectangular triangle" in which Δx is the hypotenuse, while $\Delta\sin x$ and $\Delta\cos x$ are the cathets. We observe that the true ratios $\Delta\sin x/\Delta x$ and $\Delta\cos x/\Delta x$ approach the ratios obtained from this differential triangle when Δx approaches zero. Accordingly, we get

$$\Delta\sin x \to \Delta x \cdot \cos x \quad \text{when } \Delta x \to 0, \text{ and}$$

$$\Delta\cos x \to -\Delta x \cdot \sin x \quad \text{when } \Delta x \to 0.$$

Thus

$$\frac{d \sin x}{dx} = \cos x, \qquad \frac{d \cos x}{dx} = -\sin x.$$

The trigonometric functions too are defined in the whole set R.

The derivatives of all elementary functions can be calculated on the basis of the above given derivatives, if we know some general rules holding for the construction of a derivative. These rules are the following:

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} \quad \text{(derivation of a sum)},$$

$$\frac{d(fg)}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} \quad \text{(derivation of a composite function)},$$

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg}{dx} + \frac{df}{dx}\, g(x) \quad \text{(derivation of a product)}$$

$$\frac{d}{dx}\,\frac{f(x)}{g(x)} = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{g(x)^2} \quad \text{(derivation of a ratio)},$$

$$\frac{df^{-1}}{dy} = \frac{1}{\dfrac{df}{dx}} \quad \text{(derivation of an inverse)},$$

where we have denoted $f(x) = y$ so that $x = f^{-1}(y)$. These rules could be easily derived but it would be a waste of time, since the rules have only technical significance.

To give an example of the application of these rules, let us construct the derivative of the logarithm function. Since $y = \log_a x$ means that $a^y = x$, the function $f(x) = \log_a x$ is the inverse of the function $f(x) = a^x$. The logarithm function $\log_a x$ is defined, obviously, only for non-negative real numbers. The inverse of the derivative of the latter is $1/(a^x \log a) = 1/(y \log a)$. Accordingly, using the last rule above we get

$$\frac{d \log_a y}{dy} = \frac{1}{y \log a}.$$

Of course we could then again substitute the letter x to the place of y, and write $\mathrm{d}\ \log_a x/\mathrm{d}x = 1/(x \log a)$.

In particular, for the functions e^x and $\log x$ we get, since $\log e = 1$, the derivatives

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x, \quad \frac{\mathrm{d}\log x}{\mathrm{d}x} = \frac{1}{x}.$$

If $y = \sin x$, then we write $x = \text{arc sin } y$. Thus the function $f(x) = \text{arc sin } x$ is the inverse of the function $f(x) = \sin x$. Since the inverse of the derivative of $\sin x$ is $1/\cos x = 1/(1-\sin^2 x)^{1/2} = 1/(1-y^2)^{1/2}$, we get

$$\frac{\mathrm{d}\ \text{arc sin } x}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 \leq x \leq 1.$$

In a similar way we get

$$\frac{\mathrm{d}\ \text{arc cos } x}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 \leq x \leq 1.$$

For the ratio functions

$$\text{tang } x = \frac{\sin x}{\cos x} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}$$

we get, using the rule of derivation of a ratio:

$$\frac{\mathrm{d}\ \text{tang } x}{\mathrm{d}x} = \frac{1}{\cos^2 x}, \quad \frac{\mathrm{d}\cot x}{\mathrm{d}x} = -\frac{1}{\sin^2 x}.$$

And for the inverse functions of tang x and cot x we get

$$\frac{\mathrm{d}\ \text{arc tang } x}{\mathrm{d}x} = \frac{1}{1+x^2}, \quad \frac{\mathrm{d}\ \text{arc cot } x}{\mathrm{d}x} = -\frac{1}{1+x^2}.$$

To give an example of the application of the derivation of a composite function, let us consider the derivative of the function

$$f(x) = e^{-x^2}.$$

We can consider it as a composite function by writing =

$$f(x) = e^{-x^2} = e^y, \quad \text{where } y = -x^2.$$

Thus we get

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = e^y(-2x) = -2xe^{-x^2}.$$

To give an example of the application of the derivation of a sum, let us construct the derivative of a polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

By the sum rule, and by means of the derivatives of powers we get:

$$\frac{dP(x)}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1}.$$

3 / Analytic Function

If we are given a continuous function f from $A \subset R$ to R, and if this function has a uniquely determined derivative in every point of A, the derivative also defines a function from A to R. Let us denote this function as

$$\frac{df}{dx}(x) = y.$$

This function is continuous, if $\frac{df}{dx}(x + \Delta x) - \frac{df}{dx}(x - \Delta x) \to 0$ when $\Delta x \to 0$.

If the ratio $\Delta \frac{df}{dx} \Big/ \Delta x$ approaches a uniquely determined real number when Δx approaches zero, this number is the derivative of the function $\frac{df}{dx}$ in the point x:

$$\frac{\Delta \frac{df}{dx}}{\Delta x} = \frac{\frac{df}{dx}(x+\Delta x) - \frac{df}{dx}(x)}{\Delta x} \to \frac{d\left(\frac{df}{dx}\right)}{dx}.$$

The derivative of the function $\frac{df}{dx}$ is called the second derivative of the function f, and it is then denoted as follows:

$$\frac{d^2f}{dx^2} = \frac{d\left(\frac{df}{dx}\right)}{dx}.$$

The second derivative too defines a function from R to R,

$$\frac{d^2f}{dx^2}(x) = y.$$

If this function is continuous it may have a uniquely determined derivative. This is called the third derivative of the function f and is denoted by

$$\frac{d^3f}{dx^3} = \frac{d\left(\frac{d^2f}{dx^2}\right)}{dx} .$$

In this way we can continue, and define the higher order derivatives of the given function f. If the function f is continuous and has all the derivatives $\frac{df}{dx}$, $\frac{d^2f}{dx^2}$, ... uniquely determined and continuous in an interval $A_0 \subset A$ of real numbers, this function is said to be *analytic* in the interval A_0. An analytic function $f(x)$ has the important property that it can be expanded to a series of the powers $(x-a)$, $(x-a)^2$, $(x-a)^3$, . .where a is a real number, if the derivatives of the function f in the point $x = a$ are finite.

Let us consider first a simple example of analytic function for which the expansion in power series can be given at once. This case is the case of a polynomial $P(x)$:

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

We observe that the function $P(x)$ is continuous in the whole set R of real numbers, and its all derivatives are uniquely determined and continuous also in the whole set R. For the derivatives in the point x we get, by applying successively the sum rule and the derivation of a power:

$$\frac{dP(x)}{dx} = a_1 + 2a_2x + \ldots + na_nx^{n-1},$$

$$\frac{d^2P(x)}{dx^2} = 2a_2 + 3 \cdot 2a_3x + \ldots + n(n-1)a_nx^{n-2},$$

$$\ldots\ldots$$

$$\frac{d^nP(x)}{dx^n} = n(n-1)(n-2)\ldots 1 \cdot a_n = n!a_n,$$

while all the derivatives whose order is greater than n are zero.
If we take the derivatives of $P(x)$ in the point $x = 0$ we get the a-coefficients of the polynomial, multiplied by the numbers $1, 2!, 3!, \ldots$, respectively. Thus we observe that we can write the polynomial in the form

$$P(x) = P(0) + \frac{dP}{dx}(0)\,x + \frac{1}{2!}\frac{d^2P}{dx^2}(0)\,x^2 \ldots + \frac{1}{n!}\frac{d^nP}{dx^n}(0)\,x^n.$$

Here we have the expansion of the analytic function P in terms of the powers of $x-0 = x$.

We can generalize this to an expansion of P in terms of the powers of the general difference $x—a$ by writing $x = a+(x—a)$, and rewriting the polynomial and its derivatives in terms of the powers of $x—a$. This leads to the general result

$$P(x) = P(a) + \frac{dP}{dx}(a)(x—a) + \frac{1}{2!}\frac{d^2P}{dx^2}(a)(x—a)^2 + \ldots +$$
$$+ \frac{1}{n!}\frac{d^nP}{dx^n}(a)(x—a)^n.$$

One can show that this can be generalized to apply to every analytic function f which has finite derivatives in the point $x = a$. Accordingly, we can write such an analytic function as an infinite power series

$$f(x) = f(a) + \frac{df}{dx}(a)(x—a) + \frac{1}{2!}\frac{d^2f}{dx^2}(a)(x—a)^2 + \tag{15}$$
$$+ \frac{1}{3!}\frac{d^3f}{dx^3}(a)(x—a)^3 + \ldots$$

Such an expansion is called the *Taylor expansion* of the analytic function f, and it converges in a neighbourhood of the point a.

Let us apply this to the exponential function $f(x) = e^x$. It is continuous everywhere, and has the uniquely determined derivative e^x in the point x. But then the derivative is also everywhere continuous and has the uniquely determined derivative, the second derivative of the function f, which is also e^x. Thus every derivative of the function $f(x) = e^x$ is the function f itself so that e^x is analytic everywhere. Taking their values in the point $x = 0$ we observe that every derivative of the exponential function has the value 1 in the point $x = 0$. Accordingly, the Taylor expansion of the exponential function becomes

$$e^x = 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots$$

This infinite power series represents the function e^x for every real number x.

An equally easy derivation of the Taylor expansion is obtained for the trigonometric functions $\sin x$ and $\cos x$. Since $d \sin x/dx = \cos x$, and $d \cos x/dx = —\sin x$, and since $\cos(0) = 1$ and $\sin(0) = 0$, we get at once:

$$\sin x = x—\frac{1}{3!}x^3+\frac{1}{5!}x^5—\ldots,$$

$$\cos x = 1—\frac{1}{2!}x^2+\frac{1}{4!}x^4—\frac{1}{6!}x^6+\ldots$$

The power series representing e^x, sin x, and cos x indicate obviously a connection between these three elementary functions. This connection can be made explicit if we introduce the imaginary unit i, known from school. As we remember i was defined as an algebraic element which can be added to and multiplied by real numbers, and which itself obeys the rule $i^2 = -1$. Using this rule we obtain from the above series the *Euler formula*

$$e^{ix} = \cos x + i \sin x, \tag{16}$$

which we shall need later on.

Another useful Taylor expansion is that of the function $f(x) = (1+x)^a$, where a is an arbitrary real number. For the derivatives of this function we get successively:

$$\frac{df}{dx} = a(1+x)^{a-1} \to a \quad \text{when} \quad x \to 0,$$

$$\frac{d^2f}{dx^2} = a(a-1)(1+x)^{a-2} \to a(a-1) \quad \text{when} \quad x \to 0,$$

$$\frac{d^3f}{dx^3} = a(a-1)(a-2)(1+x)^{a-3} \to a(a-1)(a-2) \quad \text{when} \quad x \to 0,$$

etc.

Accordingly the Taylor expansion at the point $x = 0$ is

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \dots$$

Here we meet again the binomial coefficients, known from school. This infinite series is called the *binomial series.*

Since the analytic functions form the most restricted type of functions, satisfying all the thinkable requirements in order to be "nice and continuous", mathematical analysis can be carried farthest to the details in the very case of analytic functions. On the other hand, the domain of applicability of relations holding true only for analytic functions is narrow.

4 / Partial and Total Derivatives

What has been said above on derivation and analytic functions can in a trivial way be generalized to functions f from $A \subset R^n$ to R. Such a function is a function of several variables $x_1, \dots, x_n$:

$$f(x_1, \dots, x_n) = y \in R, \quad (x_1, \dots, x_n) = x \in A \subset R^n.$$

The derivative of f with respect to the variable x_i can be defined as if f were a function of x_i only, i.e. keeping the other variables constant in the limiting process:

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x)}{\Delta x_i} .$$

This is called the *partial derivative* of f with respect to x_i.

It may happen that a function f does contain a variable x_i not only explicitly, like above, but also implicitly: some of the other variables may be functions of x_i. In this case the function f has the structure

$$f(x_1(x_i, \ldots), x_2(x_i, \ldots), \ldots, x_i, \ldots, x_n(x_i, \ldots)) = y \in R.$$

Then we may sometimes be interested also in the *total derivative* of f with respect to x_i. This is given by

$$\frac{Df}{Dx_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial x_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial x_i} + \ldots + \frac{\partial f}{\partial x_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial x_i} . \tag{17}$$

Accordingly, the total derivative is constructed by means of the rule of derivation of a composite function, and by the rule of derivation of a sum.

The partial derivatives of second order are defined by

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial \left(\frac{\partial f}{\partial x_j}\right)}{\partial x_k} = \frac{\partial \left(\frac{\partial f}{\partial x_k}\right)}{\partial x_j} = \frac{\partial^2 f}{\partial x_k \partial x_j}$$

For $j = k$ this is denoted

$$\frac{\partial^2 f}{\partial x_j^2} .$$

In this way we can proceed to partial derivatives of higher order and define, for instance, an analytic function of several variables. Such a function must have continuous derivatives of all orders with respect to all the variables. In a point $a \in R^n$ where all the derivatives are finite an analytic function can be expanded in power series of the differences $(x_1 - a_1), \ldots, (x_n - a_n)$. The Taylor expansion of an analytic function of several variables differs from that of an analytic function of one variable only in so far as there is a summation over the terms referring to different variables $x_1, \ldots, x_n$ in each power:

$$f(x) = f(a) + \frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \ldots + \frac{\partial f}{\partial x_n}(a)(x_n - a_n) + \tag{18}$$
$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x_1^2}(a)(x_1 - a_1)^2 + 2 \cdot \frac{1}{2!}\frac{\partial^2 f}{\partial x_1 \partial x_2}(a)(x_1 - a_1)(x_2 - a_2) + \ldots$$

5 / Integral and Integral Function

Let f be a continuous real function of one variable. Let us consider the area left between the curve $y = f(x)$ and the abscissa $y = 0$ in an interval (a,z) along the x-axis (see Fig. 9). This area $I_a(z)$ can be approximated by choosing n—1 points $x_1, \ldots, x_{n-1}$ inside the interval (a,z), and constructing the sum

$$S_a(z,n) = \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

Here $\Delta x_i = x_i - x_{i-1}$, there being $x_0 = a$ and $x_n = z$. Obviously this sum approaches $I_a(z)$ when n approaches infinity in such a way that all the intervals Δx_i approach zero:

$$S_a(z,n) \longrightarrow I_a(z) \quad \text{when } n \longrightarrow \infty \quad \text{so that all } \Delta x_i \longrightarrow 0.$$

The limit sum is called the *integral* of the function f from the point a to the point z, and it is denoted as follows:

$$I_a(z) = \int_a^z f(x)\,dx. \tag{19}$$

The function I_a as a function of z is called the *integral function* of the respective function f.

For the derivative of the integral function we get:

$$\frac{dI_a(z)}{dz} = \lim_{\Delta z \to 0} \frac{I_a(z+\Delta z) - I_a(z).}{\Delta z}$$

But $I_a(z + \Delta z) = I_a(z) + f(z)\Delta z$ so that

$$\frac{dI_a(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z)\Delta z}{\Delta z} = f(z). \tag{20}$$

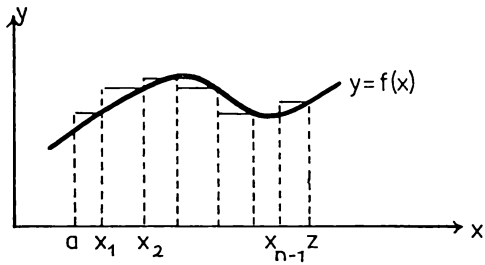


Fig. 9. For the definition of integral function.

Thus the derivative of the integral function I_a is the function f whose integral function I_a is. This gives the rule for the calculation of an area $I_a(z)$ in practice.

We observe that in order to find the area $I_a(z)$ we have first to find out a function $I(z)$ whose derivative the given function $f(z)$ is. This function contains an arbitrary additional constant, since the derivative of a constant is zero. Accordingly, we have

$$I(z) = I_0(z) + C,$$

where $I_0(z)$ is a definite function of z and C is an arbitrary real number. Then we have to choose C so that the value $I(a)$ becomes zero:

$$C = -I_0(a).$$

The desired area is then given by the value $I(z)$ so defined:

$$I_a(z) = I_0(z) - I_0(a).$$

For instance, the derivative of the function $f(x) = x^a$ is ax^{a-1} for any real number a. Accordingly, the function $f(x) = x^a$ itself is the derivative of the function $I_0(x) = x^{a+1}/(a+1)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{x^{a+1}}{a+1} = x^a.$$

Thus the area $I_0(1)$ left between the curve $y = x^a$ and the axis $y = 0$ in the interval (0,1) is given by

$$I_0(1) - I_0(0) = \frac{1^{a+1}}{a+1} - \frac{0^{a+1}}{a+1} = \frac{1}{a+1}\ .$$

If we have to calculate an area restricted by two curves $y = f(x)$ and $y = g(x)$ we simply form the difference $y = f(x) - g(x)$ and determine the area between $y = f(x) - g(x)$ and $y = 0$. So the method is not restricted to an area between a curvilinear and a linear function but can be applied more generally.

Of course all the methods of constructing the derivative, like the derivation of a composite function, sum, product, etc., can be applied in an integration problem. This gives for each rule of derivation a corresponding rule of integration, We can however, overlook this here.

6 / Space Integral

Instead of considering the integral

$$I_a(z) = \int_a^z f(x)\,\mathrm{d}x$$

as an area we can consider it as the "total mass" of the values of the function f distributed over the interval (a,z). In this way we observe that the integral can be generalized to the case of a real function of any finite number of variables.

For the sake of simplicity, let us consider the case of three real variables. If we denote the three real variables by x,y, and z, the domain of f is the vector space V_3 where any sequence (x,y,z) gives the coordinates of a certain point along three mutually orthogonal coordinate axes. Let us consider a closed volume V in this vector space. Let the volume V be connected, which means that any two points (x_1,y_1,z_1) and (x_2, y_2,z_3) within the volume V can be connected with one another by means of a continuous curve.

Let us divide the volume V into small cubes by means of planes parallel to the coordinate planes. Let the number of the cubes be n, and their volumes be $\Delta v_1, \ldots, \Delta v_n$. The limit sum

$$\int_V f(x,y,z)\mathrm{d}V = \lim_{\substack{n\to\infty \\ \Delta v_l \to 0}} \sum_{l=1}^{n} f(x_l,y_l,z_l)\Delta v_l$$

if it exists and is finite, is called the integral of the function f over the volume V. Here (x_l,y_l,z_l) is a point within the volume Δv_l.

If the volume V fulfills certain conditions of regularity the volume integration of a continuous function f over V can be performed by means of successive integrations with respect to one of the variables x,y, and z in turn. For this purpose let us assume that x_1 is the minimal and x_2 the maximal x-coordinate of the points within the volume V. And let us assume that any straight line in the (xy)-plane, parallel to the coordinate axis y, meets the boundary of the projection of V on this plane only at two points whose y-coordinates are given by the two functions $y = y_1(x)$ and $y = y_2(x)$, with $y_2(x) \geq y_1(x)$. And finally, let us assume that any straight line parallel to the z-axis meets the boundary of the volume V only at two points whose z-coordinates are given by the two functions $z = z_1(x,y)$ and $z = z_2(x,y)$, there being $z_2(x,y) \geq z_1(x,y)$.

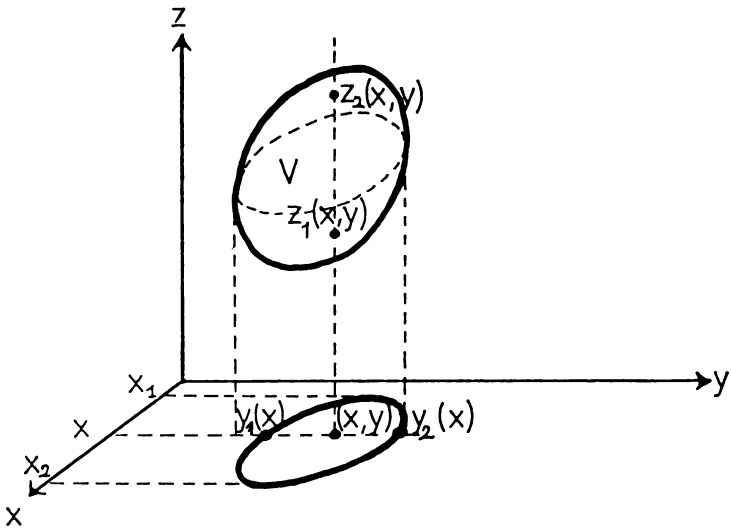


Fig. 10. Integration over a volume

On these conditions we can perform the integration over the volume V as a three-fold integration

$$\int_V f(x,y,z)\mathrm{d}V = \int_{x_1}^{x_2} \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z.$$

This means that we perform first the integration of f as a function of one variable, viz. z, from the value $z_1(x,y)$ to the value $z_2(x,y)$. As a result we get a function F of x and y:

$$F(x,y) = \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z.$$

Then we perform the integration of $F(x,y)$ as a function of one variable, viz. y, from the value $y_1(x)$ to the value $y_2(x)$. This gives us a function G of x:

$$G(x) = \int_{y_1(x)}^{y_2(x)} F(x,y)\mathrm{d}y.$$

Finally, we perform the integration of $G(x)$ from x_1 to x_2 to get

$$\int_V f(x,y,z)\mathrm{d}V = \int_{x_1}^{x_2} G(x)\mathrm{d}x.$$

In order to get simple functions F and G as "*integrands*" we may perform transformations of the variables x,y, and z. If x,y, and z are represented as three functions of some new real variables ξ, η, and ζ,

$$x = x(\xi, \eta, \zeta),$$
$$y = y(\xi, \eta, \zeta),$$
$$z = z(\xi, \eta, \zeta),$$

so that the correspondence $(x,y,z) \longleftrightarrow (\xi, \eta, \zeta)$ is one-to one, the functions being continuous and having first partial derivatives, then we can transform the integral as follows:

$$(21) \qquad \int_V f(x,y,z)\mathrm{d}V = \int_{\xi_1}^{\xi_2} \mathrm{d}\xi \int_{\eta_1(\xi)}^{\eta_2(\xi)} \mathrm{d}\eta \int_{\zeta_1(\xi,\eta)}^{\zeta_2(\xi,\eta)} f(x(\xi,\eta,\zeta),\ y(\xi,\eta,\zeta),$$
$$, z(\xi,\eta,\zeta)) \left| \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} \right| \mathrm{d}\zeta.$$

Here appears the *functional determinant* defined by

$$\left| \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{vmatrix}$$

Let us calculate an example. Let there be

$$f(x,y,z) = (x^2 + y^2)z,$$
$$x_1 = -1,\ x_2 = +1,$$
$$y_1 = -\sqrt{1-x^2},\ y_2 = +\sqrt{1-x^2},$$
$$z_1 = 0,\ z_2 = h.$$

Accordingly, the volume V is a cylinder of the height h, whose projection on the (xy)-plane is given by the circle $x^2 + y^2 = 1$.

First we get

$$F(x,y) = \int_0^h (x^2+y^2)z\mathrm{d}z = \frac{1}{2}(x^2+y^2)h^2.$$

Then we introduce in the (xy)-plane the new variables r and φ by

$$x = r\cos\varphi,\ y = r\sin\varphi.$$

We have thus

$$F(x(r,\varphi),y(r,\varphi)) = \frac{1}{2}h^2r^2,$$

$$\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r.$$

The integration over the circle $x^2+y^2 = 1$ means integration from 0 to 1 in the r-variable, and integration from 0 to 2π in the φ-variable. Accordingly,

$$\int_V (x^2+y^2)z\,dV = \frac{h^2}{2}\int_0^1 r^3dr \int_0^{2\pi} d\varphi = \frac{h^2}{2}\cdot\frac{1}{4}\cdot 2\pi = \frac{\pi h^2}{4}.$$

If the boundaries of V are given by curves indicating constant values of the variables of integration, then the order of successive integrations is arbitrary.

The definition of space integral can in a trivial way be generalized to any finite number of variables.

5 § *Topological Notions*

The more advanced analysis of the real functions, and of the complex functions closely associated with them, is based on topological notions and the notion of complex number. We shall discuss elements of these notions briefly in the remaining sections of this chapter.

1 / Topological Space

Topology is a generalization from the 'geometrical' properties of real numbers. Representing the real numbers along a straight line, and cutting the line at two points, say a and b, we have an 'interval' (a,b) of real numbers (see Fig. 11). Since the real numbers form a continuum,

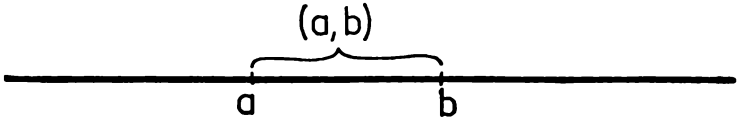


Fig. 11. An interval (a,b) in the geometrical representation of real numbers.

there is an infinite number of real numbers within each interval (a,b) in so far as $a \neq b$. In other words, we cannot make the interval (a,b) so small that there would be less than an infinite number of real numbers between the numbers a and b, as far as $a \neq b$. Omitting the points a and b themselves from the interval (a,b) we still have an infinite number of points in the so obtained 'open interval' formed by the points between a and b. When the end points a and b are counted as belonging to the interval (a,b) we have a 'closed interval'. The notions of open and closed interval are generalized in topology by introducing in a general set the operation of closure in the following way.

Let W be a set, and $F(W)$ the set of its subsets. Let the operation of closure, $\mathbf{C}$, be defined as a function from $F(W)$ to $F(W)$ obeying the following rules:

1° $\mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$ for any set $A \subset W$,

2° $\mathbf{C}(A_1 \cup A_2) = \mathbf{C}(A_1) \cup \mathbf{C}(A_2)$ for any two sets $A_1, A_2 \subset W$, and

3° $\mathbf{C}(\{a\}) = \{a\}$ for any set of W containing only one element $a \in W$.

A set W where an operation of closure is in this way defined is called a *topological space*.

In a topological space one uses the following terms: Every element $a \in W$ is called a *point* of the space W. The set $\mathbf{C}(A) \subset W$ is called the *closure* of the set $A \subset W$. A set $A \subset W$ which is identical with its closure, $\mathbf{C}(A) = A$, is called a *closed set*. A set A whose complement is a closed set, $\mathbf{C}(W-A) = W-A$ (the complement $W-A$ of A is the set of all the points of W which do not belong to A), is called an *open set*. Any open set A to which a point a belongs, is called a *neighbourhood* of the point a.

Of course, an example of topological space is the set R of real numbers. Any subset A of R is composed of the intervals of the type (a,b) we studied above. We can define the closure $\mathbf{C}(A)$ as the set obtained from A when both the end points a and b of all the intervals (a,b), $a \neq b$, belonging to A, and all the single points (a,b), $a = b$, belonging to A are counted with. Any single point and any set $A \subset R$ composed of mere single points and closed intervals is a closed set. The open sets of R are those composed of open intervals. We call this natural topology of real numbers the *Euclidean topology*.

The Euclidean topology can be obviously defined in every cartesian product space $R \times R \times \ldots \times R = R^n$ composed of the space R of the real numbers. This makes every space R^n a topological space having

the Euclidean topology. In the cartesian product space R^n there is a point for every sequence $(x_1,x_2,\ldots,x_n)$ of n real numbers. The correspondence between the points of R^n and such sequencies is one-to-one. We can define an addition of any two points

$$x = (x_1,x_2,\ldots,x_n) \in R^n \quad \text{and} \quad y = (y_1,y_2,\ldots,y_n) \in R^n$$

by the vector addition rule:

$$x + y = (x_1 + y_1,\ x_2 + y_2,\ldots,x_n + y_n) \in R^n.$$

We can also define the multiplication of a point $x = (x_1,x_2,\ldots,x_n) \in R^n$ by a real number $k \in R$ by the scalar multiplication rule:

$$kx = (kx_1,kx_2,\ldots,kx_n) \in R^n.$$

In this way R^n becomes an n-dimensional vector space.

We can introduce to the space R^n a metric, whether positive-definite, or negative-definite, or indefinite, in an infinite number of different ways. We always can, if we like, introduce to R^n even the Euclidean metric. This is done by defining an inner product between $x \in R^n$ and $y \in R^n$ by

$$< x,y > = x_1y_1 + x_2y_2 + \ldots + x_ny_n.$$

Then the norm and the distance become Euclidean:

$$|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

$$|x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2}.$$

Therefore, every space R^n can be called Euclidezable: an Euclidean metric can be always defined in it, if we like, but sometimes we may need a non-Euclidean, or even non-positive-definite metric in an Euclidezable space. What is essentially expressed by saying that the space R^n is Euclidezable is 1° that it has the Euclidean topology, and 2° that, if we like, an Euclidean metric can be defined in it. To distinguish R^n from the n-dimensional Euclidean space E_n, where an Euclidean metric is actually defined, we thus call R^n an *n-dimensional Euclidezable space*. Obviously, every vector space V_n over the field R of real numbers is an n-dimensional Euclidezable space.

2 / Topological Mappings

Let W_1 and W_2 be two topological spaces. Let h be a function from W_1 to W_2, and H the corresponding function from $F(W_1)$ to $F(W_2)$ induced by h. Then h is called a *homeomorphism* from W_1 to W_2, if

1° h is one-to-one, and

2° the operation of closure is invariant in H: $H\mathbf{o}\mathbf{C} = \mathbf{C}\mathrm{o}H$ or $(H\mathbf{o}\mathbf{C})(A) = H(\mathbf{C}(A)) = \mathbf{C}(H(A)) = (\mathbf{C}\mathrm{o}H)(A)$ for any $A \subset W_1$.

The homeomorphisms thus are the one-to-one functions that transform open sets to open sets, and closed sets to closed sets.

A *continuous mapping* h is a function from W_1 to W_2 obeying the following less stringent condition:

$$H(\mathbf{C}(A)) \subset \mathbf{C}(H(A)) \quad \text{for any } A \subset W_1.$$

In the case of continuous mapping we can say that if $H(A)$ is a closed set, then A is closed, and if $H(A)$ is an open set, then A is open.

Homeomorphism could be defined also as a one-to-one function h, when both h and h^{-1} define a continuous mapping. The homeomorphisms and the continuous mappings can both be called topological mappings.

3 / Topological Manifolds

If a neighbourhood A of a point a of a topological space W can be homeomorphically mapped to an open set $X(A)$ of an Euclidezable space R^n, the topological space W is *locally Euclidezable* at the point a. Let x be the homeomorphism from A to $X(A)$. The function x associates then to each point b of the neighbourhood A n real numbers

$$x_1(b), x_2(b), \ldots, x_n(b),$$

viz. the coordinates of the point $x(b) \in R^n$:

$$b \xrightarrow{x} (x_1(b), x_2(b), \ldots, x_n(b)) = x(b) \in R^n \quad \text{for every } b \in A.$$

These real numbers are called the *local coordinates* of the point b introduced by the homeomorphism x.

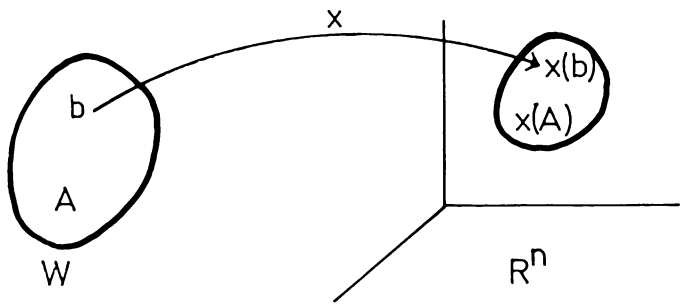


Fig. 12. The local coordinates

If a topological space W is locally Euclidezable everywhere, with respect to a given n-dimensional space R^n, and if W can be covered by an enumerable union $A_1 \cup A_2 \cup A_3 \cup \ldots$ of open sets of W, we call W an n-dimensional *topological manifold*, and denote it by M_n. In a topological manifold we can introduce everywhere local coordinates $(x_1(b), x_2(b), \ldots, x_n(b)) \in R^n$ which are valid for the points b in some neighbourhood A of any fixed point $a \in W$, but we cannot introduce the full system of Euclidean coordinates: there need not be one-to-one mapping of the whole space W to the whole Euclidezable space R^n. An example of such a 2-dimensional topological manifold is a sphere.

Let $A_1 \cup A_2 \cup A_3 \cup \ldots$ be a covering of a topological manifold M_n by open sets. Let A be one of the sets of the covering, and x be a homeomorphism from A to $X(A) \subset R^n$. If f is a one-to-one function from $X(A)$ to $Y \subset R^n$, then the composite function $f \circ x = y$ defines a homeomorphism from A to $Y \subset R^n$. The two local coordinate systems defined in A by x and by y are related to one another by the one-to-one transformation f:

$$y(b) = f(x(b) \quad \text{for every} \quad b \in A \subset M_n.$$

Such a function f is called a *transformation of coordinates*. Let the function f be m-times differentiable, (with continuous derivatives, with respect to each of its arguments). If we allow the local coordinates of A be defined either by the original coordinates x or by any new coordinates obtained from the original ones by an m-times differentiable function f, we have in the set A a differentiable coordinate system of the class m. Let us assume that we have in each set belonging to the covering $A_1 \cup A_2 \cup \ldots$ of M_n a differentiable coordinate system of the same class m.

Let A and B be two arbitrary sets belonging to such a covering of M_n. If then all the allowed coordinate systems in $A \cap B$ are connected with one another by m-times differentiable functions, the manifold M_n is called a *differentiable manifold* of the class m. A differentiable manifold whose class of differentiability is that of $m = \infty$, is called *analytic manifold*.

Recently, a new branch of algebraic topology was created by A. Grothendieck. The spaces of Grothendieck are expected to have great significance in future topological cybernetics (cf. p. 8), but we shall not discuss them here.

6 § Complex Numbers

1 / What are the Complex Numbers for a Science?

It is easy to understand why the rational and even the real numbers in general are helpful in every science in which measurement is performed. To speak first of the rational numbers, every result of measurement is a rational number indicating the proportion of the measured magnitude to a unit of measurement. As to the necessity of the real numbers in general, we know proportions which can be observed in reality but which cannot be expressed in terms of rational numbers. Such a proportion is, for instance, the proportion of the diagonal of a square to the length of a side of the same square. In addition to the rational numbers also the irrational numbers, and thus all the real numbers, are needed in science based on observation and measurement.

In order to see the necessity of the complex numbers in such a science we must consider an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0, \quad a_n \neq 0. \tag{22}$$

Such kind of equations, where the coefficients $a_0, \ldots, a_n$ are all real numbers, often appear in the analysis of the results obtained ny observation and measurement. We may need to solve the equation for the unknown x in this analysis. Then we meet a restriction of the real numbers: we may have an equation of the above type which does not have any solution x given by a real number.

An example of the equation of the kind mentioned is

$$x^2 + 1 = 0.$$

Since this equation means that $x^2 = -1$ and since there is no real number x such that its square would be equal to -1, the equation has no solution by real numbers.

Here the complex numbers come to help the analyst. One can show that every equation of the above type, where the highest power of x is x^n, has exactly n solutions given by complex numbers $x = c_1, \ldots, x = c_n$. Accordingly, we can rewrite the equation by means of these solutions in the form

$$a_n(x-c_1) \cdot \cdot (x-c_n) = 0. \tag{23}$$

Some of the solutions c_l may be multiple solutions which means that we must take a power $(x-c_l)^{k_l}$ of the binomial $(x-c_l)$ in the left side

of this equation, k_l being a positive integer which shows the multiplicity of the solution $x = c_l$. We call $c_1, \ldots, c_n$ the 'roots' of the equation in question.

Of course the complex numbers must again disappear from the analysis before we can compare the results with the reality. This may happen, for instance, in a way that we have to deal in our analysis only with such sums $c_l + c_k$ or such products $c_l c_k$ of the roots which happen to be real numbers. However, we need knowledge of the roots of the real coefficient algebraic equations in order to be able to operate with them in our analysis.

Here, as always, the usefulness of a mathematical tool in science is not restricted to the case in which it was first needed but was soon extended to new problems. Once we have introduced the complex numbers in a science in connection with the real-coefficient equations we observe that they are also useful in many other connections. This is actually the case. However, we shall restrict the discussion here to the role of complex numbers as roots of real-coefficient algebraic equations.

2 / The Algebraic Operations on Complex Numbers

We know from school that the complex numbers are introduced by starting with the equation $x^2 + 1 = 0$. We enlargen the field R of real numbers to a set C which contains as its elements all the combinations of the form

$$c = a + ib.$$

Here a and b are real numbers while i is an element of C defined by

$$i = \sqrt{-1}.$$

So we have $i^2 = -1$, and likewise $(-i)^2 = -1$, so that i and $-i$ are the roots of the equation $x^2 + 1 = 0$.

Let us now consider the set

$$C = \{c = a + ib;\ a \in R,\ b \in R,\ i = \sqrt{-1}\}.$$

We can represent the elements of this set by the points in a coordinate plane (xy) where a is the x-coordinate and b the y-coordinate of the point c. We observe at once that when a and b are allowed to go through all real values the point c goes through all the points of the coordinate plane. Obviously, the correspondence between the complex numbers c and the points of the plane R^2 is one-to-one. Accordingly, the operations

on complex numbers can be studied on this coordinate plane, remembering, that with the y-coordinates we must associate the imaginary unit i as factor in order to get complex numbers.

First we observe that the plane (xy) is a two-dimensional vector space. Thus the addition of vectors as well as the scalar multiplication of vectors can be performed on the points c. We get simply

$$c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2), \text{ and}$$

$$kc = ka + ikb,$$

respectively. The former operation defines an addition for the complex numbers, and the latter defines a multiplication of the complex numbers by a real number k.

To define a multiplication for complex numbers we can introduce the coordinates (r,φ) instead of the original (x,y) by writing

$$a = r\cos\varphi, \quad b = r\sin\varphi$$

for each point c (See Fig. 13).

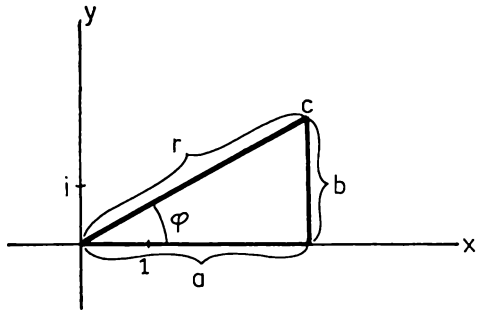


Fig. 13. The complex plane

Then we can write

(24) $$c = a + ib = r(\cos\varphi + i\sin\varphi) = re^{i\varphi},$$

where we have applied the Euler equation (see p. 64). Now the multiplication of complex numbers can be defined simply by

$$c_1c_2 = r_1e^{i\varphi_1} \cdot r_2e^{i\varphi_2} = r_1r_2e^{i(\varphi_1+\varphi_2)}.$$

Thus two complex numbers are multiplied with one another by multiplying their 'absolute values' r_1 and r_2, and adding up simultaneously their 'arguments' φ_1 and φ_2.

It is easy to verify also that the multiplication just defined obeys associativity, commutativity, and distributivity with respect to addition. We also observe at once that each complex number c has the inverse

$$c^{-1} = r^{-1}e^{-i\varphi}.$$

A further operation is defined for complex numbers, viz. the *complex conjugation*. This is a function from C to C itself, which associates with each complex number c its 'complex conjugate'

$$c^* = a - ib = re^{-i\varphi}.$$

This operation is important because for each c the sum $c + c^*$ and the product cc^* are real numbers:

$$c + c^* = 2a, \quad cc^* = r^2 \geq 0. \tag{25}$$

Complex conjugation means geometrically a reflection of the points c with respect to the x-axis so that c and c^* are symmetrically with respect to this axis.

3 / The Roots of Algebraic Equations

When we know that every algebraic equation

$$a_n x^n + \ldots + a_1 x + a_0 = 0, \quad a_n \neq 0,$$

has exactly n solutions, or roots, given by some complex numbers $x = c_1, \ldots, x = c_n$, we can easily derive either these roots themselves or at least some of the properties they have. We restrict ourselves in the following only to the case of real coefficients $a_0, \ldots, a_n$.

The above equation expresses the equality of two (in general) complex numbers, viz. the number $a_n x^n + \ldots + a_1 x + a_0$ and the number 0. Taking the complex conjugate of both of these numbers we have another equality, since the complex conjugates of two identical complex numbers are the same:

$$a_n x^{*n} + \ldots a_1 x^* + a_0 = 0.$$

In this equation the real coefficients are the same as before, and only x is replaced by x^*. Accordingly, this equation has exactly the same solutions as before:

$$x^* = c_1, \ldots, \quad x^* = c_n.$$

We conclude that our original equation has for each root $x = c_i$ also the root $x = c_i^*$. Thus the roots $c_1, \ldots, c_n$ may contain single real numbers or pairs (c, c^*) of mutually conjugate complex numbers.

Let us now consider our equation in some special cases. For $n = 1$ we have the equation

$$a_1x + a_0 = 0, \quad a_1 \neq 0.$$

This equation has one root and it is real:

$$x = -a_0/a_1.$$

For $n = 2$ we have the equation

$$a_2x^2 + a_1x + a_0 = 0, \quad a_2 \neq 0.$$

We know that it has exactly two roots, viz. a number c and its conjugate c^*. Writing the equation by means of these roots it reads

$$a_2(x-c)(x-c^*) = 0.$$

Expanding this we get

$$a_2x^2-a_2(c + c^*)x + a_2cc^* = 0.$$

By comparing the coefficients with the original form of the equation we find that

$$a_1 = -a_2(c + c^*) \text{ and } a_0 = a_2cc^*.$$

Here are two equations for the determination of the two roots c and c^*. We get

$$c + c^* = 2a = -a_1/a_2 \text{ so that } a = -\frac{a_1}{2a_2}, \text{ and}$$

$$cc^* = r^2 = a_0/a_2.$$

Since $r^2 = a^2 + b^2$ we have the further result

$$b^2 = r^2-a^2 = \frac{a_0}{a_2} - \frac{a_1^2}{4a_2^2} = \frac{4a_0a_2-a_1^2}{4a_2^2}$$

The two roots are thus given by

$$x = -\frac{a_1}{2a_2} \pm \frac{\sqrt{a_1^2-4a_0a_2}}{2a_2}$$

For the equation beginning with the power x^3 we at once obtain the result that it has either one real root and two mutually conjugated complex roots, or then all its roots are real. These are obviously the only possibilities to accord with the general rule that for each root c the complex conjugate c^* is also a root. This fact can be used in the solution of such an equation. In a similar way we can use our general knowledge on the roots of algebraic equations in the solution of equations containing higher powers of x.

We may also use the possibility to represent the complex number c in the form $re^{i\varphi}$ when solving algebraic equations. As an example let us consider the equation

$$x^n = a_0,$$

where a_0 is a real number, either positive or negative. If $c = re^{i\varphi}$ has to be a root we must have

$$r^n e^{in\varphi} = r^n(\cos n\varphi + i \sin n\varphi) = a_0.$$

Since a_0 is real this gives two equations, viz.

$$r^n \cos n\varphi = a_0 \quad \text{and} \quad \sin n\varphi = 0.$$

The latter gives

$$n\varphi = 0, \pi, 2\pi, 3\pi, \ldots,$$

or

$$\varphi = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{(2n-1)\pi}{n},$$

as possible different solutions for φ. For these values of φ we have only two possible values of $\cos n\varphi$, viz.

$$\cos n\varphi = +1 \quad \text{if } \varphi = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(2n-2)\pi}{n},$$

$$\cos n\varphi = -1 \quad \text{if } \varphi = \frac{\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{(2n-1)\pi}{n}.$$

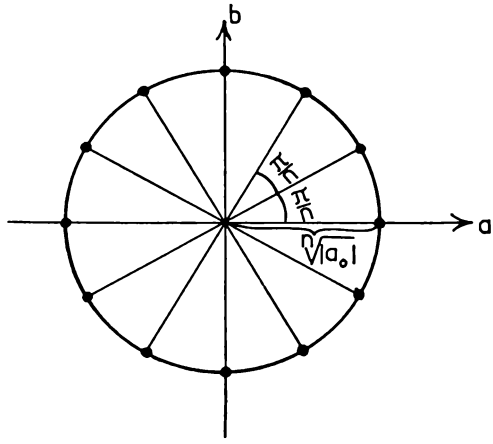


Fig. 14. The roots of $x^n = a_0$ in the complex plane.

If a_0 is positive we obviously have to choose the upper sequence of n values for φ. If a_0 is negative we have to choose the lower n values for φ. These values together with the value

$$r = \sqrt[n]{|a_0|}$$

complete the solution of the equation $x^n = a_0$. The solutions can be represented graphically in the complex plane: they are points localed symmetrically along the circle whose center is the origin and whose radius is $\sqrt[n]{|a_0|}$.

CHAPTER II:

Fundamental Cybernetic Notions

1 § A Glance at the History of Cybernetics

1 / Descartes

The analogy between animal and machine was first pointed out by René Descartes, the well-known 17th century French philosopher.

Descartes saw the animal as a machine which is in interaction with its environment. The environment acts on the animal through the stimuli *S* which are received by the animal. The animal acts on the environment through its reactions *R* by which the animal responds to the stimuli. Thus a notion which is much employed in behavioral science, viz. the *stimulus-response connection S—R*, comes from Descartes. So also does the term *reflex*. According to him a reaction *R* to a stimulus *S* comes from the animal like a reflexion from a mirror. He had thus a fully deterministic conception of the behaviour of the animal.

As is well known Descartes did not apply the analogy between animal and machine to man. At that time when religion still controlled intellectual life it would indeed have been too daring an enterprise to relate the entities of human consciousness with material things. Descartes gave to the thoughts, the emotions and the will of the human being an independent existence, and so landed in a dualistic conception of reality.

It is paradoxical that a philosopher who introduced the first cybernetic conceptions became generally known as a philosopher of dualism. In fact this is really unjust: Descartes did not "invent" dualism — he just gave a formulation to the philosophical ideas which were typical

and common in the European thinking in that century. In the atmosphere saturated by religion it had hardly been possible to extend the materialistic conceptions of cybernetics to the entities of human consciousness.

2 / Pavlov

The posthumous reputation of many a great innovator is often narrow in an unjust way. Ivan Pavlov, for instance, is generally known for his experimental work with reflexes, while the great theoretical framework created by Pavlov has been less known outside of Soviet Union. However, there are good reasons to see in Pavlov the founder of cybernetic thinking.

The main part of Pavlov's scientific work concerned the analogy between animal and machine in a way which now would be called cybernetics. One part of this work was, of course, the study of reflexes. One can easily understand the interest in reflexes in the works of the early pioneers of cybernetics: these are the simplest and most obvious mechanisms in the behavior of animal which act like machines. We know that Pavlov developed the theory of reflexes. He shoved that except for the unconditioned reflexes which need not and cannot be learned there are conditioned reflexes which are formed by learning. In addition to the 'first signal system' associated with elementary conditioned reflexes there is the 'second signal system' associated, for instance, with the learning of language.

However, Pavlov also developed fundamental notions related to the cybernetic analogy between animals and machines. He was the first to approach the study of living organisms from a system theoretical point of view. The notion of 'system' is loaned from theoretical physics. There we have both 'closed systems' and 'open systems'. A physical system composed of material things is called closed if it is not in interaction (in exchange of energy) with the environment. The system is open, if there is interaction between the system and the environment.

In the theoretical framework of Pavlov the reflexes commit interaction between a living organism and the environment. Accordingly, an organism must be characterized as an open system. Now comes the decivise question: what is then the fundamental behavioral difference between a living organism and a lifeless open physical system? Pavlov was the first who answered the question by introducing the central notion of cybernetics: the notion to which one has referred by the terms *self-regulation* (this was the term used by Pavlov), or *self-control*, or *self-*

steering. Pavlov[2] stated as early as 1917 that the behaviour of a living organism is distinguished from all the other open material systems by its highly developed ability of self-regulation, or self-steering.

Thus one can take the notion of self-steering as a Pavlovian notion. The introduction of the notion of self-steering to the analysis of living organisms was the beginning of cybernetic thinking in the modern sense of the word.

3 / Wiener

When the analogy between animal and machine was further developed it was ever more obvious that the notion of self-steering, introduced by Pavlov, was to be essential in the analogy. The study of self-steering material systems is the foundation of the science which is in our time called cybernetics, according to the well known book of Norbert Wiener[3].

The scientific work of Wiener has been helpful in co-ordinating a great variety of different approaches under a common title and in an integrated science. When co-ordinating the various approaches the significance of *feedback* as an essential condition of self-steering was revealed better than ever before. So one can say that the indication of the general role of feedback couplings in all the self-steering systems was one of the most important contributions of Wiener to cybernetic theory.

4 / Mc Culloch and von Neumann

The early developments in the theory of automata were one of the approaches from which cybernetics emerged. M.A. Turing had as early as 1936 formulated an interesting thesis associated with automata. The most important contributions to cybernetics from the theory of automata perhaps came from two scientists, both of whom are distinguished by their large scale of scientific interest and ability. I am referring here to Warren McCulloch, a mathematician who became a professor of psychiatry, and Johann von Neumann, one of the great mathematicians of our century.

2. Ivan Pavlov, *Selected Works*. Moscow (no year of printing).
3. N. Wiener, Cybernetics, 1. ed. 1948, 2. ed. New York 1961.

In particular the materialistic investigation of living beings in terms of cybernetic notions was developed by McCulloch and von Neumann. McCulloch, together with the mathematician Pitts, proved in 1943 the equivalence of *neural nets* and finite automata[4]. The significance of this result is very great in the understanding of the nervous system. McCulloch and Pitts also developed interesting ideas on the materialistic interpretation of universals in cybernetic terms[5]. One can also mention an interpretation of the will given by McCulloch[6].

As an outstanding mathematician Johann von Neumann gave a valuable contribution to the discussion of the possibilities of cybernetic theory in the explanation of creative intellectual performance[7]. Like another famous mathematician, A.N. Kolmogorov, he recommended an open point of view according to which it would not be wise to see any principal restrictions in these possibilities. Von Neumann also suggested a cybernetic interpretation of self-reproduction[8].

The many technological contributions of von Neumann, may be passed by in this short review of the development of the main ideas of cybernetics.

5 / Oskar Lange

If Pavlov was the first to formulate the idea of purposive self-steering, Wiener the one who directed general attention to the significance of feedback as a causal basis of self-steering, and if McCulloch and von Neumann have contributed most to the cybernetic analysis of the phenomena of life, then Oskar Lange must be mentioned as the social scientist who has contributed very much to the introduction of cybernetic method in social science.

Lange emphasized the significance of cybernetic method in connection with two general problems, viz. that of 'wholes' and that of dialectical development[9]. Lange also gave an elegant mathematical formulation for the connection between feedback and self-steering.

4. W. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophysics* 5, s. 115—133, 1943.
5. W. Pitts and W. McCulloch, How we know universals — the perception of auditory and visual forms. *Bull Math. Biophysics* 9, s. 124—147, 1947.
6. W. McCulloch, Why the mind is in the head? In Jeffress (Ed.), *Cerebral Mechanisms of Behavior*, Hafner Co 1951, 2. Ed. 1967.

To conclude this short review of the development of cybernetic ideas it should be mentioned that there are of course a lot of important works of a technological or mathematical nature which we have passed by above. The technical construction of many an ingenious apparatus has contributed to the understanding of cybernetics. The mathematical work done in the theory of differential equations and in the study of ergodic processes in operational analysis and elsewhere has also contributed to cybernetic theory.

2 § System Notions

1 / General Systems

Let $V_1, \ldots, V_n$ be any n sets of objects of reality. Any subset S of the cartesian product $V_1 \times \ldots \times V_n$,

$$S \subset V_1 \times \ldots \times V_n, \tag{1}$$

defines a *general system*. The sets $V_1, \ldots, V_n$ are called the system objects, or the objects of the system S.

The above definition of general system is very general indeed. It identifies the notion of system and the notion of mathematical relation between n objects of reality. Indeed, a subset S is composed of some of the sequences $(x_1, \ldots, x_n)$, where x_1 is an element of V_1, x_2 an element of V_2 etc.:

$$S = \{(x_1, \ldots, x_n);\ x_1 \in V_1, \ldots, x_n \in V_n\}.$$

Such a subset of $V_1 \times \ldots \times V_n$ was called — as can be seen on page 34, an n-member mathematical relation.

The definition of general system does not contain any specification of the nature of the objects of reality concerned. Neither does it contain

7. See, for instance, J. von Neumann, The general and logical theory of automata. In L.A. Jeffress (Ed.), *Cerebral Mechanisms in Behaviour*, 1. ed. 1950, 2 ed. Hafner Co. 1967.
8. J. von Neumann, *Theory of Self-Reproducing Automata*. London 1966.
9. Oscar Lange, *Wholes and Parts*, Pergamon Press 1965 (the Polish original in Warsaw 1962), and
Oscar Lange, *Theory of Reproduction and Accumulation*, Pergamon Press 1969 (the Polish original in 1965).

any guarantee of the significance of the relation S: whether it expresses a complete, sufficient or insufficient characterization of the mutual relations of the objects in question for some particular purpose or not, whether the relation is invariant over some interval of time or not, etc.

For instance, the set V_1 could be composed of the national account statistics in Austria in 1954, containing as elements the national income of Austria in that year, the number of people between 15 and 64 in Austria in that year, the number of tons of wheat produced in Austria in 1954, etc. The set V_2 could be composed of the properties of a 1963 Citroën model, containing as elements the number of persons for which the model is drawn up, the number of litres of petrol the car uses per one hundred kilometres, etc. The set V_n could contain as elements the distance of the moon from the earth on September 21, 1947, the number of sunspots observed on March 3, 1921, and other astronomical observations referring to various days and years. Still any subset S composed of the sequences $(x_1, \ldots, x_n)$ is a general system.

We can see that the definition of general system has very little to give to the analysis of reality. Just a mathematical relation between some objects of reality has no significance whatsoever, unless the objects to which it refers form a kind of reasonable whole localized in space and time, and unless the relation itself has some properties of invariance and significance. We meet here the fact which we have met earlier already: generalization for the sake of generalization itself is not meaningful in mathematical analysis. It is the specification which gives sense to mathematical relations.

In other words: if we would define our object of study as a general system, and call it 'system' then one and the same system would contain elements from very different kinds of material objects, and the same material object would involve an infinite number of different 'systems' (cf. W. R. Ashby, *An Introduction to Cybernetics*, London 1956, p. 39). Here we want to give a more definite content to the notion of system.

2 / Material Systems

We specify general systems first by requiring that the system objects $V_1, V_2, \ldots, V_n$ belong to a *material object M localized in space and time.* The volume of space where the object M is localized may be a connected piece of space, as it is if M is a certain chair in a certain room, or a certain car driving in a certain street, or a certain man having a certain

name, or a certain country, etc. Or the volume where M is localized may be a disconnected one, i.e. composed of several connected pieces. In this case we say that M is a collection of material objects $M_1, M_2, \ldots, M_k$, each of which is localized in a connected volume of space. Examples of a material collection M are the four chairs located in a certain room, or a certain ten cars driving in a certain street, or a group of people sitting around a certain table, or a population of all the citizens in a given country in a given year, etc.

The localization of a material object, or of a material collection, is not rigid. A chair can be moved to another room and it is still the same chair. A car driving in the street may drive to another city, or to another country, and it is still the same car. The citizens of a given country may travel through the world without losing their citizenship. Thus we cannot require any fixed location: the only thing we can do is to require that the object M to which all the system objects $V_1, \ldots, V_n$ refer is a real material object located in some connected or disconnected volume of space at every moment of some interval of time.

At every moment of some interval of time? Indeed it is meaningless to extend the consideration to any point of time whatsoever. A material object may be transformed to something which does not interest us anymore in our study. For instance, a material thing used for some particular purpose, like a chair, may be broken and cast away. If a material object is a living being it dies some day, and is transformed to a collection of unliving material objects which has other properties than the living being had, and thus is characterized by other kind of relations than the living being was. If the relation R of our system has to express something characteristic of the material object M, then we must obviously restrict somehow the interval of time to which our system refers. In other words, we must require that M is localized not only in space but also in time.

We already referred to a requirement being imposed on the relation R, defining the general system (former denoted by S) which we are studying. In order that our notion of system be helpful in the analysis of reality the relation R must obviously give a *sufficient characterization* of the properties of the material object M. This condition is important even though it is difficult to say in any general terms which characterization is sufficient in each case. But it is obvious that we must attempt at a sufficient characterization of all the relevant properties of M when we are constructing the relation R, if the notion of system has to have any significance at all in the analysis of reality. Except that R must

give a sufficient characterization, relevant to the purpose of the study, this relation must also be *invariant over some interval of time* in order to have significance in the analysis of reality. Thus we have two requirements to be imposed on the basic relation of the system: sufficient characterization of M and invariance over some interval of time. We call such a relation R a *characteristic relation of the material object* M, and indicate this by writing

$$R = \text{Ch } M.$$

We may sometimes consider systems whose material object M or some of its elements are capable of reproducing themselves. Such a material object is, for instance, a living organism, or the human population of a given country. We must then specify in the definition of our system whether the descendants of M are to be included in M or not. The solution depends on whether the system relation R is a characteristic relation of the descendants too or not. For instance, a human being and his descendants are individually different in many essential aspects. It is hardly meaningful to treat them but as different material objects. On the other hand, the people from which the population of a country is composed are coupled with one another by social institutions which in the main may remain the same from one generation to another. Even though social revolutions may change the structure of institutions radically, the social institutions between social revolutions at least may be invariant enough so that it is meaningsful to consider them, during some time as a socio-economic system undergoing changes but referring to the same material object. This object contains the population of the country in question (and, in addition to this, for instance, the tools of labour in this country). In this case it is meaningful to include the descendants of M to the material object M of our system.

We can now define a *material system* S as a combination

(2) $\qquad S = (M,R) \quad \text{where } R = \text{Ch } M.$

In words: a material system is a combination of a material object M and a relation R such that R is a characteristic relation of M.

Here R alone requires to define a general system. R is a subset of the cartesian product $V_1 \times \ldots \times V_n$ in which $V_1, \ldots, V_n$ are the system objects. The system objects all refer here to the same material object M.

M may be located in space either in a connected or a disconnected volume, and this localization is not fixed in either case to a given closed volume of space. If the localization is disconnected we may speak sometimes of a material collection M instead of a material object M.

M is always localized in a certain interval of time. M may or may not include its own descendants: this must be specified separately in each case.

R must give a sufficient characterization of all the relevant properties of M, and it must also be invariant over some interval of time.

3 / Systems of Definite Topology

A material object may or may not have a *material boundary surface.* Indeed we can consider as a material object, in the physical sense of the word, any piece of matter, i.e. any collection of molecules. This collection of molecules may contain molecules of different substances, of different gases or liquids or solid substances, etc. We may consider, for instance, all the matter within a certain volume of space. The surface of this volume may cut a chair, and close within the volume a part of this chair together with a part of the air around the chair. All the molecules within the part of the chair which is included in the volume, together with all the molecules of the air included in the volume form the material object we are considering. Such a material object, composed of a part of the chair and of a part of the air, has no material boundary surface.

However, there are in reality many material boundary surfaces which separate in a natural way material objects from other material objects. A chair, for instance, has a material boundary surface, and water in a glass has a material boundary surface. The skin forms the material boundary surface of man.

From now on we restrict the consideration to such material systems whose material object M_b has a material boundary surface. Accordingly, we are studying a system $S = (M_b,R)$ where $R = \text{Ch } M_b$. If M_b fills a connected volume in space, this boundary surface is connected. If M_b is a collection of objects, and fills a disconnected volume in space, its boundary surface is composed of several distinct parts. For instance, the material boundary surface of the population of a given country is composed of all the skins (and other material surfaces) of all the people of that country. Just as the localization of a material object is not fixed to a given closed volume of space, the boundary surface of an object is not fixed to any rigid position in space. Of course the boundary surface of a man moves and is deformed all the time the man is moving. The boundary surface of the population of a certain country may even develop new disconnected parts: this happens when

new people are being born, and an infant is separated from the mother's body.

The existence of boundary surface means that the material system in question has a definite topological structure in space-time. We can call such material systems *systems of definite topology*. More exactly, we understand by a system of definite topology a material system having a material boundary surface, connected or disconnected, through which the interaction of the system with the environment, as well as the mutual interaction of the distinct parts of the system, is localized in space and time.

The topological structure of systems can be expected to become an important part of cybernetic theory in future, topological cybernetics (cf. p. 8). Irrespectively of this, a definite topological structure in space-time must be considered as an important defining attribute of all the systems discussed in cybernetic theory, whether elementary or not.

Material systems which have a material boundary surface can be divided into two classes, i.e. to those systems whose boundary surface is completely isolating, and to those whose boundary surface is only relatively isolating. The former systems are completely isolated by their surface from the rest of the material world, there being no interaction whatsoever between the system and the environment. Such systems are in physics called *closed systems*. An example is a collection of gas molecules closed within a rigid box which cannot be deformed mechanically or otherwise and whose walls do not let in or out any form of energy (heat, or electricity, etc.) If M_b is such a material object, the system objects $V_1, \ldots, V_n$ are properties of the collection of molecules within the box, for instance, temperature, pressure, entropy, specific gravity, colour etc. of the gas. The system relation R expresses the physical laws which are valid between temperature, pressure etc. in such a gas.

There is a very important law which is valid for all closed systems, viz.

The Law of Entropy: In a closed system the physical entropy increases monotonically, i.e. all the structural and functional organization within the system disappears in the course of time.

We shall later on study the exact definition of physical entropy. Now a general characterization of what happens when entropy increases will suffice.

For a gas closed in an isolating box the increase of entropy means the following. Whatever be the original distribution of molecules over the volume, and whatever be the original distribution of the velocities

of the molecules, these distributions tend to a homogeneous one when the system is isolated from the rest of the world. In other words, the system tends toward an end where each component of the gas is homogeneously distributed over the volume of the box, and the molecules of each component are moving by the same velocity homogeneously in all directions. For a liquid or a solid state body which forms a closed system the Law of Entropy means that all the spatial structures formed by the molecules, for instance, the crystals, disappear and the matter within the closed space assumes a gas-like state having the properties of homogeneity mentioned above.

Of course a closed system is an idealization which is never exactly realized in nature. Most material systems which have a material boundary surface are only relatively isolated from the rest of the world. They are 'open systems' of a particular kind, i.e. *relatively isolated systems.* The term was introduced by the Polish cybernetician Greniewski[10].

A closed or completely isolated system

A relatively isolated system

Fig. 15. Material systems having a (connected) material boundary surface

For instance, a collection of gas molecules closed within a box becomes a relatively isolated system if the walls of the box can be pressed so that the volume of the box changes, or if the walls let heat or electricity or other forms of energy in and out. If M_b is this kind of collection of molecules, then the system objects $V_1, \ldots, V_n$, representing the properties of the gas like its temperature, volume, pressure, entropy etc., may indicate the influences of the environment on the system and the

10. Henryk Greniewski, *Cybernetics Without Mathematics*, Pergamon Press 1960.

influences of the system on the environment through the walls of the box. The system relation R indicates laws holding between temperature, volume, pressure etc. under such conditions.

For a relatively isolated system the Law of Entropy is not necessarily valid. In other words,

> the structural and functional organization may increase in a relatively isolated system.

This is a fundamental property of relatively isolated systems, and we shall later on study in detail how the organization indeed may increase in the particular kind of relatively isolated system which we shall call cybernetic.

4 / Cybernetic Systems

As pointed out by Oskar Lange[11], "dialectical materialism asserts the existence of material systems, the elements of which are linked by a chain of cause-and-effect relations".

It is thus in accord with the general program of dialectical materialism to assume material systems whose interaction with the environment obeys a cause-and-effect relation. Obviously such systems are always relatively isolated. The specification of cause and effect in the interaction of a relatively isolated system with the environment gives us the notion of *cybernetic system*. This specification can be done as follows.

Let (M_b,R) be a material system which has a relatively isolating material boundary surface. We specify the system relation $R \subset V_1 \times \ldots \times V_n$ by dividing the system objects to two factors,

$$(3) \qquad \begin{aligned} X &= V_1 \times \ldots \times V_m, \quad \text{and} \\ Y &= V_{m+1} \times \ldots \times V_n, \end{aligned}$$

where $V_1, \ldots, V_m$ are the *cause objects* and $V_{m+1}, \ldots, V_n$ are the *effect objects*.

The distinction between the cause and effect objects cannot be performed on any formal grounds but must be based on the recognition of the real conditions under which the interaction of the system with the environment happens. The causes $V_1, \ldots, V_m$ indicate the influence of the environment on the system, while the effects $V_{m+1}, \ldots, V_n$ indicate the influence of the system on the environment.

11. Oskar Lange, *Wholes and Parts*, p. 1, Pergamon Press 1965.

When the system objects $V_1, \ldots, V_n$ of a relatively isolated system have been divided into causes and effects we say that *cybernetic causality* is explained for the system. Such a system will be called cybernetic in this book. In such a system the system relation can be expressed as a relation between the total cause X and the total effect Y:

(4) $$R_c \subset X \times Y.$$

When we speak of a *causal relation* in the cybernetic sense we refer to such a relation R_c in this book. A *cybernetic system* can now be defined as a combination

(5) $$\boxed{S = (M_b, R_c) \text{ where } R_c = \text{Ch } M_b.}$$

The relation R_c represents the *fundamental causal recursion* defined in the cybernetic system S.

For instance, a collection of gas molecules closed in a box which has relatively isolating walls becomes a cybernetic system as soon as we distinguish the system objects, which indicate the influence of the environment on the system, from those objects by which the system acts on the environment. If we press the box, then the change ΔV of the volume of the box indicates the influence of the environment on the system, while the response of the system to the environment can be transmitted, for instance, by means of the resulting increase Δp in gas pressure or by the increase ΔT of gas temperature. Thus we would have in this case

$$X = \{V\} \quad \text{and} \quad Y = \{p\} \times \{T\},$$

where $\{V\}$, $\{p\}$, and $\{T\}$ are the sets of the possible values ("states") of volume, pressure, and temperature, respectively. The relation R_c between X and Y would in this case be (for an ideal gas)

$$\frac{pV}{T} = \text{const.},$$

when written as the rule by which the elements of $R_c \subset X \times Y$ are chosen (for the expression of a relation by such a rule, see p. 33).

On the other hand, if we bring heat into the system the resulting increase ΔT of gas temperature would now indicate the influence of the environment on the system. The subsequent increase ΔV of gas volume and the increase Δp of gas pressure would now transmit the system's response to the environment. Accordingly, in this case we

would have $X = \{T\}$ and $Y = \{p\} \times \{V\}$. We observe that the distinction between cause and effect cannot be performed on any formal grounds but only on the basis of the analysis of the real sequence of events in each particular case. The formal rule expressing the relation R_c would still have the same form, viz. $pV/T = \text{const.}$

The elements of X will be called the *input states* of the system, and the elements of Y the *output states* of the system. Using these terms the causal interaction of a cybernetic system with the environment can be described in the following way. The environment acts on the system

> "by inducing in it certain states of a strictly defined type such as temperature, pressure, electric charge, feeling, sense impression" (Lange, ibid., p. 4) — the combinations of these states are the input states of the system.

On the other hand, the system acts on the environment

> "by assuming certain states of a strictly defined character, e.g. temperature, magnetic field, colour, generation of sounds, motion" (Lange, ibid., p. 4) — the combinations of these states are the output states of the system.

We can indicate the division of the system objects $V_1, \ldots, V_n$ to the *input objects* $V_1, \ldots, V_m$ (the cause objects) and the *output objects* $V_{m+1}, \ldots, V_n$ (the effect objects) graphically by letting the former be represented by "channels" coming to the system S, and the latter by "channels" departing from the system S (see Fig. 16).

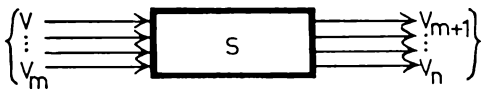


Fig. 16. The division of the system objects in a cybernetic system

In a cybernetic system we can always — if we like — define even a third kind of state concept, viz. the inner states of the system. This definition is based on the following theorem:

A relation $R_c \subset X \times Y$ can always be parametrized by a set $\mathfrak{P} = \{f_s; s \in \Sigma\}$ of functions from X to Y, i.e. expressed as the union

$$R_c = \bigcup_{s \in \Sigma} f_s. \tag{6}$$

Slint for proof: Take an element $y_0 \in Y$ for which the cardinality of the set $\{(x,y_0); x \in X, y_0 \in Y, y_0 \text{ fixed}\}$ in the relation R is maximal. Introduce the index set Σ_0 by writing $\{(x,y_0); x \in X\} = \{(x_s,y_0); s \in \Sigma_0\}$. Obviously this index set gives the minimal index set for the paramet-

rization $\mathfrak{P}$, and a number of parametrizations can be easily constructed by using this index set.

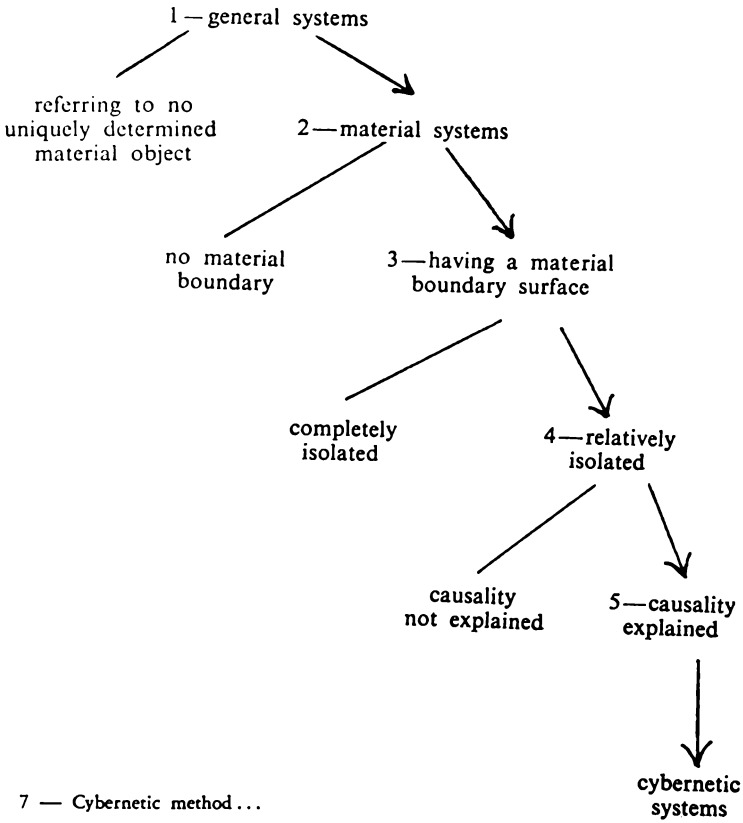
The indices s in a parametrization $\mathfrak{P}$ of R_c are called the *inner states* of the cybernetic system in question. Different parametrizations thus define different sets of inner states for the system. The function

(7) $\qquad f_s : X \to Y$

is the *state function*, or the *behavior function* of the system in the inner state s. It indicates how the system in the inner state s responds to a given stimulus, i.e. what is the output of the system for a given input:

$$f_s(x) = y, \quad x \in X, \quad y \in Y.$$

To sum up, we have defined the cybernetic system by using the following tree of successive definitions:



1: general system is a relation $R \subset V_1 \times \ldots \times V_n$ where the system objects $V_1, \ldots, V_n$ are sets of real objects,

2: material systems are general systems whose system objects refer to a uniquely determined material object M whose characteristic relation the system relation is: $R = \text{Ch } M$,

21: a uniquely determined material object is a definite collection of matter localized in space and time,

211: material object is considered localized if it is within a certain volume of space at every moment of time within a given interval of time, whereby

2111: the volume of localization may be either connected, or composed of several mutually disconnected parts, and

211: the surface of the volume of localization is not fixed but can move in the course of time,

212: it is to be decided in each case separately whether the descendants of a material object, capable of reproducing itself, are to be included in the system or not,

22: the system relation is a characteristic relation of the material object M if

221: it gives a sufficient characterization of the properties of M for the particular purpose of the study of the system, and

222: is invariant over a sufficient long interval of time to be useful in the study,

3: a material system has a material boundary surface if its surface is composed of material elements different from the neighboring elements inside and outside the system so that *all* the energetic impulses from the environment or from the system do not penetrate the boundary; a material system having a boundary surface, connected or disconnected, defined in space-time and indicating the movements, the descendants etc. of the system and of its parts, was said to have a definite topology in space-time,

4: a material system M_b having a definite topology is relatively isolated if there is some form of interaction between the system and the environment through the boundary surface, expressed in terms of the system objects $V_1, \ldots, V_n$,

5: cybernetic causality is explained for a relatively isolated system if the system objects $V_1, \ldots, V_n$ are divided into causes and effects, i.e. objects by which the environment acts on the system and by which the system responds to the environment: this is not a formal distinction but one which must be based on the analysis of the real sequence

of events in each particular case; if the system is composed of several disconnected parts (cf.211), then the mutual interaction of the parts is not considered as an interaction between the system and the environment but as an interaction occuring *within the system* itself.

In the context of a cybernetic system we use certain terms which were defined as follows:

input states: the elements of $X = V_1 \times \ldots V_m$, where $V_1, \ldots, V_m$ are the system objects representing the causes,

output states: the elements of $Y = V_{m+1} \times \ldots \times V_n$, where $V_{m+1}, \ldots, V_n$ are the system objects representing the effects,

causal relation: the system relation of a cybernetic system when expressed as $R_c \subset X \times Y$,

inner states: the elements of Σ, when $\mathfrak{P} = \{f_s; s \in \Sigma\}$ gives a parametrization of the system relation R_c so that $R_c = \bigcup_{s \in \Sigma} f_s$,

state function: each of the functions f_s, determining the outputs of the system in the inner state s by the equation $f_s(x) = y$, for $x \in X$, $y \in Y$; the state function can also be called the *behavior function.*

It is not the notion of general system but the notion of cybernetic system which is fruitful in the analysis of reality. In short, a *cybernetic system is both material, topological, and causal.*

5 / Explicit Introduction of Time in a Cybernetic System

Let it be still emphasized that the causality of which we are speaking cannot be defined by any formal process but must be concluded on the basis of an analysis of the sequence of real events in each case. This is important to observe, since one often in West tends to define causality as a formal time-ordering.

To emphasize that causality in the sense of dialectical materialism is not merely a formal time-ordering of events on whatever arbitrarily chosen ground, time was not at all explicitly introduced in the above consideration. We spoke of the system objects $V_1, \ldots, V_n$ and of their division to the causes $V_1, \ldots, V_m$ and the effects $V_{m+1}, \ldots, V_n$ without mentioning time. Of course the notion of time was implicitly involved in our consideration in so far as what we call a cause always precedes in time what we call effect. Let us now consider the explicit introduction of time to a system.

Let us begin by introducing time to a general system $R \subset V_1 \times \ldots \times V_n$. We associate first with such a system a *calendar* K defined as a subset

of the set T of all the moments of time: $K \subset T$. Each general system thus has its calendar, i.e. a particular set of the moments of time. The set T of all time points can here be mapped in a one-to-one way to the set R of all real numbers: for each real number there is one point of time, the real number in question being the indicator of time in some unit of time. Accordingly, the calendar K of a general system R can always be mapped in a one-to-one way to a set composed of real numbers. These real numbers may form a continuum or they may form a discrete, or even finite set. We say that the system has then a continuum, discrete, or finite calendar, respectively.

We define the interval $K_t^{t'}$ in the calendar K as the set of the moments t'' of time which obey $t \leq t'' < t'$ and belong to K:

$$K_t^{t'} = \{t''; \ t \leq t'' < t', \ t'' \in K\}.$$

In particular, $K_t^t = \{t\}$ indicates a single time point t.

We are now able to introduce time explicitly to the system objects $V_1, \ldots, V_n$. Let us introduce n abstract sets, viz. $A_1, \ldots, A_n$, one for each system object. Let v_i be a function from K to the set A_i. Let us denote by $v_i|K_t^{t'}$ the restriction of the function v_i to the interval $K_t^{t'}$. We call such a restriction the *segment of the function* v_i associated with the interval $K_t^{t'}$. In particular, the restriction $v_i|K_t^t$ is an element of A_i when associated with the point t of time. We call it the *momentary value* of the function v_i at the time point t, and denote it by $v_i(t)$:

$$v_i(t) = v_i|K_t^t.$$

Now we define the system object V_i as the set of all the segments $v_i|K_t^{t'}$ composed of all the functions v_i from K to A_i:

$$V_i = F\{v_i|K_t^{t'}; \quad v_i : K \to A_i, \quad K_t^{t'} \subset K\}.$$

By this we have defined V_i as the set of all the function segments that can be composed by restricting all the functions from K to A_i to all the intervals $K_t^{t'} \subset K$. The elements of the system objects $V_1, \ldots, V_n$ thus are such segments of functions. The elements of different system objects are here by no means restricted to refer to one and the same interval $K_t^{t'}$: they may well refer to different intervals.

When we consider a cybernetic system $S = (M_b, R_c)$ instead of a general system R we must introduce time so that all the cause objects refer to the same interval of time, the same being true of all the effect objects. This can be performed in the following way. Let x be a function from the calendar K of the system to the cartesian product $A_1 \times \ldots \times A_m$, where $A_1, \ldots, A_m$ are the abstract sets associated with the cause objects

$V_1, \ldots, V_m$ of the system. Let $x|K_t'$ be the restriction of x to the interval K_t'. The set X of all the possible input states of the system is then defined in a time-explicit form by

$$X = \{x|K_t';\ x: K \to A_1 \times \ldots \times A_m,\ K_t' \subset K\}. \tag{8}$$

Thus the set of all the possible input states is the set of all the function segments which can be composed by restricting all the functions x from K to $A_1 \times \ldots \times A_m$ to all the segments K_t' of K.

The time-explicit form of the set Y of all the possible output states can be defined in a similar way:

$$Y = \{y|K_t';\ y: K \to A_{m+1} \times \ldots \times A_n,\ K_t' \subset K\}. \tag{9}$$

However, the explicit introduction of time to a cybernetic system is not completed hereby. We must still specify the time dependence of the cause-and-effect chains contained in the causal relation $R_c \subset X \times Y$.

The cause-and-effect chains in a cybernetic system are based on the analysis of the sequence of events which occur in material reality. Of course the effect comes in reality always after the cause. This is the starting point in the explicit introduction of time to the causal relation R_c.

Let us consider an effect associated with a definite moment of time, i.e. the *momentary output*

$$y(t) = y|K_t'.$$

If we denote by K^t the part of the calendar which contains all the calendar time before t, i.e.

$$K^t = \{t';\ t' < t,\ t' \in K\},$$

then the cause of $y(t)$ certainly appeared during the interval K^t. This is all what we can say as a general rule on the explicit introduction of time to the causal relation $R_c \subset X \times Y$.

If we make some assumptions on the state descriptions Σ of our system we can say more. For this purpose let us consider a system $S^t = (M_b, R_c^t)$ obtained from the original system $S = (M_b, R_c)$ by restricting the original calendar K to K^t. Let X^t and Y^t be the corresponding restrictions of the sets X and Y of the input and output states. In the restricted system the causal relation is given by

$$R_c^t \subset X^t \times Y^t.$$

Let us introduce a parametrization $\mathfrak{P}^t$ of R_c^t, and let us denote the corresponding state description by

$$\Sigma^t = \{s(t)\}.$$

When we let t run through the calendar K, we get a set $\{S^t;\ t \in K\}$ of restricted systems, and a set $\{\Sigma^t;\ t \in K\}$ of state descriptions referring to different points of time.

If the state descriptions referring to the different points of time are such that the state $s(t')$ is uniquely determined by the state $s(t)$ at an earlier time t and by the input segment $x|K_t^{t'}$,

$$s(t') = g(s(t),\ x|K_t^{t'}),$$

and the momentary output $y(t)$ is uniquely determined by the inner state $s(t)$,

$$y(t) = f'(s(t)),$$

the cybernetic system S is called *state-determined.* The function g is the *state-transition function* of the system S, and the function f' is its *output function.*

Using the state-transition function we can rewrite the output function as follows:

$$y(t') = f'(s(t')) = f'(g(s(t),\ x|K_t^{t'})) = f(s(t),\ x|K_t^{t'}).$$

It is usually given in the latter form. Then the state-transition function g and the output function f are both defined in the cartesian product $\Sigma^t \times X_t^{t'}$ where $X_t^{t'}$ is composed of the restrictions of all the x's to the interval $K_t^{t'}$ of calendar. We can then write.:

$$\begin{aligned} &g\colon\ \Sigma^t \times X_t^{t'} \to \Sigma^{t'} && \text{(state-transition),} \\ &f\colon\ \Sigma^t \times X_t^{t'} \to Y^{t'} && \text{(output function).} \end{aligned} \tag{10}$$

Let it be added that the dependence of the functions g and f on the segment $x|K_t^{t'}$ may imply *both* the dependence on the values $x(\tau)$ of the function x within the interval $K_t^{t'}$ *and* the dependence on the end points t and t' of this interval. Thus the state-transition and the output functions may be given by

$$\begin{aligned} s(t') &= g(s(t),\ \{x(\tau)\},\ t,t') \quad \text{where } \tau \in K_t^{t'}, \\ y(t') &= f(s(t),\ \{x(\tau)\},\ t,t') \quad \text{where } \tau \in K_t^{t'}. \end{aligned}$$

In this case the system S is called *'non-time-invariant'*.
If there is no explicit dependence on t and t', i.e. if

$$\begin{aligned} s(t') &= g(s(t),\ \{x(\tau)\}) \quad \text{where } \tau \in K_t^{t'}, \\ y(t') &= f(s(t),\ \{x(\tau)\}) \quad \text{where } \tau \in K_t^{t'}, \end{aligned}$$

then the system S is called *time-invariant.* We shall meet later examples of both time invariant and non-time-invariant cybernetic systems.

In the above formulae it was assumed that $t' > t$, in which case the

state-transition and the output functions express the state and the output at the time t' in terms of the state and the inputs associated with earlier points of time. The state description of the system is then called *non-anticipatory*. Obviously, we can consider the state-transition and the output functions in a non-anticipatory state description as a time-explicit expression of causality in a state-determined cybernetic system.

However, one should emphasize that the *existence* of cybernetic causality has nothing to do with the state description of the system. The existence of causality concerns the distinction of the system objects $V_1, \ldots, V_n$ to input and output objects, and is based on the real sequence of events occuring in material reality. A cybernetic, and thus causal, system is completely defined even without any introduction of inner states at all. The introduction of the inner states is a purely formal procedure, and so also is the choice of a non-anticipatory state description.

One can sometimes also introduce an *anticipatory* state description to cybernetic system. The state description is called anticipatory, if it associates some outputs $y(t')$ with inputs $x(t)$ occurring at a later point of time, $t > t'$. Such a case occurs in a state-determined system if the state-transition and the output relations remain formally valid even for $t' < t$. We shall give later examples of both non-anticipatory and anticipatory state descriptions in a (causal) cybernetic system[12].

6 / Digital Systems

The explicit introduction of time to a cybernetic system, as explained above, consisted of

(1) the introduction of a calendar K, of
(2) the definition of the input and output states $x \in X, y \in Y$, as function segments in this calendar, and of
(3) the specification of time in the cause-and-effect chains of the system.

12. It should be mentioned that in mathematical systems theory it has been customary to introduce several notions of 'causality', some of which are defined by a purely formal process. For instance, M.D. Mesarovic (Mathematical theory of general systems and some economic problems, in Kuhn-Szegö (Ed.), *Mathematical Systems Theory and Economics*, Springer-Verlaf 1969) distinguishes between 'external', 'internal', and 'time causality'. However, the two latter are defined purely formally and thus do not represent causality as a category explaining material reality. The first category corresponds to the cybernetic causality as explained here.

As an example of this procedure we shall now consider the explicit introduction of time to a particular kind of cybernetic systems, viz. the digital systems. By a digital system we understand a cybernetic system $S = (M_b, R_c)$ where all the interaction between the system and the environment as well as between all the different parts of the system happens in the form of short, countable impulses. Examples of such impulses are the beats of heart and the nervous impulses exchanged between the neurons in a nervous system. It is essential that we can distinguish the successive impulses from one another, count them, and associate each of them with a definite moment of time. Accordingly, the calendar of a digital system is composed of an enumerative set of discrete points of time. We can thus always represent the calendar K of a digital system by a set of successive integers,

$$K = \{t_0,\ t_0 + 1,\ t_0 + 2,\ \ldots,\ t_n\}.$$

Here t_n may or may not be $+\infty$.

Once the calendar of a digital system is introduced we can easily complete the introduction of time to such a system. An interval $K_t^{t'} = \{t'';\ t \leq t'' < t',\ t'' \in K\}$ now reduces to the set

$$K_t^{t'} = \{t,\ t + 1,\ t + 2,\ \ldots,\ t' - 1\} \subset K$$

of successive points of time. An input segment $x|K_t^{t'}$ thus reduces to the set

$$x|K_t^{t'} = \{x(t),\ x(t + 1),\ \ldots,\ x(t' - 1)\}$$

of momentary inputs. In a similar way an output segment $y|K_t^{t'}$ reduces to the set

$$y|K_t^{t'} = \{y(t),\ y(t + 1),\ \ldots,\ y(t' - 1)\}$$

of momentary outputs. The set X of all input states is defined, as before, as the set of all the possible input segments:

$$X = \{x|K_t^{t'};\ x : K_t^{t'} \to A_1 \times \ldots \times A_m,\ K_t^{t'} \subset K\}.$$

In a similar way the set Y of all output states is defined as the set of all the possible output segments:

$$Y = \{y|K_t^{t'};\ y : K_t^{t'} \to A_{m+1} \times \ldots \times A_n,\ K_t^{t'} \subset K\}.$$

We will remember that $A_1, \ldots, A_m$ are the sets of objects associated with the system objects $V_1, \ldots, V_m$ (the cause or input objects), and $A_{m+1}, \ldots, A_n$ are the sets of objects associated with the system objects $V_{m+1}, \ldots, V_n$ (the effect or output objects). These sets may be any sets of real objects whatever.

Let us now consider a state-determined digital system. The conditions of state-determinacy read now, when written for two successive points of time, t and $t' = t + 1$, as follows:

$$(11)\quad \begin{aligned} s(t+1) &= g(s(t),\ x\big|K_t^{t+1}) = g\ (s(t), x(t), t) \quad \text{(state-transition)},\\ y(t+1) &= f(s(t),\ x\big|K_t^{t+1}) = f\ (s(t), x(t), t) \quad \text{(output function)}, \end{aligned}$$

for a non-time-invariant system, and

$$(12)\quad \begin{aligned} s(t+1) &= g(s(t),\ x\big|K_t^{t+1}) = g\ (s(t), x(t)) \quad \text{(state-transition)},\\ y(t+1) &= f(s(t),\ x\big|K_t^{t+1}) = f\ (s(t), x(t)) \quad \text{(output function)}, \end{aligned}$$

for a time-invariant system. A time-invariant state-determined digital system is called an *automaton* in the customary sense. However, we could also speak of non-time-invariant automata as automata which are capable of changing itself. We shall meet such kinds of changing automata later.

In particular, a time-invariant state-determined digital system is called a *finite automaton*, if the system objects $A_1, \ldots, A_n$ are all finite sets. Let the number of elements in the sets $A_1, \ldots, A_n$ be $l_1, \ldots, l_n$, respectively. Then the number of elements in the total cause object $A_1 \times \ldots \times A_m$ is $l_1 l_2 \ldots l_m = N_1$ and the number of elements in the total effect object $A_{m+1} \times \ldots \times A_n$ is $l_{m+1} l_{m+2} \ldots l_n = N_2$:

$$\# A_1 \times \ldots \times A_m = l_1 l_2 \ldots l_m = N_1,$$

$$\# A_{m+1} \times \ldots \times A_n = l_{m+1} l_{m+2} \ldots l_n = N_2.$$

Both N_1 and N_2 are finite numbers. Accordingly, each input function x may have only N_1 different values, and each output function y only N_2 different values. These values may be numbered by the integers $1, 2, \ldots, N_1$ and $1, 2, \ldots, N_2$, respectively. So we get for the maximal ranges of x and y:

$$\max R_x = \{x_1,\ x_2, \ldots, x_{N_1}\},$$

$$\max R_y = \{y_1,\ y_2, \ldots, y_{N_2}\}.$$

Accordingly, in a finite automaton the ranges of input and output functions are finite sets.

We can employ the finiteness of the ranges of input and output functions in a finite automaton to the *binarization of its inputs and outputs.* By this we mean a decomposition of each possible input value x_k and of each possible output value y_k into a set of input or output components each of which can have only two possible values, viz. 0 or 1. This can be performed simply by expressing the index k as a binary number

in terms of the numbers 0 and 1. For instance, from the seven first input values $x_1, \ldots, x_7$ we get in this way three binary input components each of which can have either the value 0 or the value 1 in the following way:

Original input values:	Binary components: Input I:	Input II:	Input III:
x_1	0	0	1
x_2	0	1	0
x_3	0	1	1
x_4	1	0	0
x_5	1	0	1
x_6	1	1	0
x_7	1	1	1

Thus a momentary input $x(t)$ which can have the seven values $x_1, \ldots, x_7$ can be decomposed into three simultaneous binary inputs $x_{\mathrm{I}}(t)$, $x_{\mathrm{II}}(t)$, and $x_{\mathrm{III}}(t)$, each of which can only have the value 0 or the value 1. We can indicate this graphically by drawing to our system as many input channels as there are different components of binary inputs. The output can be treated in a similar way to get a finite number of binary output channels.

Let us consider as examples some very simple finite automata where the inputs and the outputs have been binarized. The conjunction element, the disjunction element, the byrocrat, and the multiplier (see Fig. 17) are all finite automata which have only one and constant inner state. Thus there is no state-transition in these automata, and their output functions are represented simply by

$$y(t+1) = f(x(t)),$$

where the output function f is different for different automata. Representing both $y(t+1)$ and $x(t)$ in binary form, the functioning of these simple automata is completely characterized by indicating the transitions $x(t) \rightarrow y(t+1)$ as given in Fig. 17. The binary input or output 0 can be interpreted to represent the phenomenon 'nothing happens in the channel', while the input or output 1 indicates an impulse in the channel.

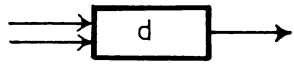
A somewhat more complicated finite automaton is the "memory element" M which has two inner states s_1 and s_2, and the state-transition and output functions given in Fig. 18.

A Conjunction Element c:

$x(t) \longrightarrow y(t+1)$:

$00 \longrightarrow 0$
$01 \longrightarrow 0$
$10 \longrightarrow 0$
$11 \longrightarrow 1$

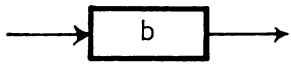
A Disjunction Element d:



$x(t) \longrightarrow y(t+1)$:

$00 \longrightarrow 0$
$01 \longrightarrow 1$
$10 \longrightarrow 1$
$11 \longrightarrow 1$

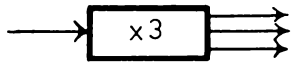
A Byrocrat b:



$x(t) \longrightarrow y(t+1)$:

$0 \longrightarrow 0$
$1 \longrightarrow 1$

A Multiplier $\times$ 3:

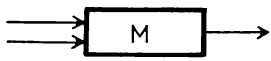


$x(t) \longrightarrow y(t+1)$:

$0 \longrightarrow 000$
$1 \longrightarrow 111$

Fig. 17. Four one-state finite automata

The Memory Element M:



$s(t), \dot{x}(t) \longrightarrow s(t+1)$:	$s(t), \dot{x}(t) \longrightarrow y(t+1)$:
$s_1, 00 \longrightarrow s_1$	$s_1, 00 \longrightarrow 0$
$s_1, 01 \longrightarrow s_1$	$s_1, 01 \longrightarrow 0$
$s_1, 10 \longrightarrow s_1$	$s_1, 10 \longrightarrow 0$
$s_1, 11 \longrightarrow s_2$	$s_1, 11 \longrightarrow 0$
$s_2, 00 \longrightarrow s_1$	$s_2, 00 \longrightarrow 0$
$s_2, 01 \longrightarrow s_1$	$s_2, 01 \longrightarrow 1$
$s_2, 10 \longrightarrow s_1$	$s_2, 10 \longrightarrow 1$
$s_2, 11 \longrightarrow s_1$	$s_2, 11 \longrightarrow 1$

Fig. 18. A two-state finite automaton

From the five types of finite automata represented in Fig. 17 and 18 we can construct the conditioned reflex automaton shown in Fig. 19. The coupling of the elements in this system follows the rule: if an element S_i has an output channel which leads to another element S_k (as one of its input channels), then the output of S_i in that channel at time t is simultaneously an input of the element S_k. The binarized inputs and outputs are to be interpreted so that the input or output value 0 means 'nothing happens in the channel' while the input or output value 1 means 'an impulse in the channel'.

We can easily see that the finite automaton of Fig. 19 indeed can learn a simple conditioned reflex. For this purpose let us perform five experiments with the automaton.

Experiment 1. Let the memory element M of the automaton be in the inner state s_1. Let an impulse come to he automaton along the input channel *UCS* at time t. At time $t+1$ it is multiplied to two impulses in the multiplier $\times 2$. One of these impulses dies in the memory element at time $t+2$ without changing the inner state s_1 (to change it one should need two simultaneous impulses to M). The other impulse continues from the first byrocrat at time $t+2$, and from the second byrocrat at time $t+3$, goes through the disjunction element, and is in the reaction channel R at time $t+4$. Thus, the automaton has performed an unconditioned reflex *UCS—R*.

Experiment 2. Let the memory element M be still in the inner state s_1. Let an impulse come to the system along the input channel *CS* at time t. At time $t+1$ it is multiplied to two impulses in the multiplier. One of them again dies in M at time $t+2$ without changing the inner state s_1. The other continues from the first byrocrat at time $t+2$ but dies at time $t+3$ in the conjunction element c. There is no reaction in the channel R. The automaton could not do the reflex *CS—R*.

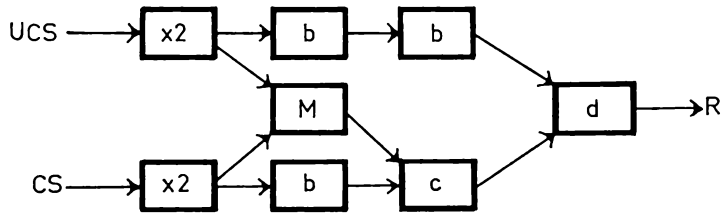


Fig. 19. A time-invariant conditioned reflex automaton

Experiment 3. Let the memory element be still in the inner state s_1. Let two simultaneous impulses come to the system at time t, one along the channel *UCS* and one along the channel *CS*. Both of them are multiplied to two impulses at time $t+1$ in the respective multipliers. At time $t+1$ the memory element thus receives simultaneously two impulses which change the inner state: the new inner state is s_2. A memory trace is thus born as a consequence of reinforcement. At time $t+2$ both of the impulses which entered *M* die, because *M* can react by an output only if it already was beforehand in the inner state s_2. The remaining two impulses continue from the first byrocrats at time $t+2$. The one which entered the system along the channel *CS* dies at time $t+3$ in the conjunction element. The other continues at time $t+3$ from the second byrocrat, and at time $t+4$ from the disjunction element, and is thus in the channel *R* at time $t+4$. The automaton has performed the unconditioned reflex *UCS—R*, and learned the conditioned reflex *CS—R*.

Experiment 4. Let the memory element now be in the inner state s_2, as a result of the conditioning performed in Experiment 3. Let an impulse come to the automaton along the channel *CS* at time t. At time $t+1$ it is multiplied to two impulses in the multiplier. One of them continues from the memory element *M* at time $t+2$ transferring this element back to the inner state s_1. The other impulse continues at time $t+2$ from the byrocrat. Thus the conjunction element *c* receives at time $t+2$ two impulses, and accordingly sends out an impulse at time $t+3$. This impulse continues from the disjunction element, and is in the channel *R* at time $t+4$. The automaton has performed the conditioned reflex *CS* which it learned in Experiment 3.

Experiment 5. After Experiment 4 the memory element is again in the inner state s_1. Accordingly it has forgotten the conditioned reflex *CS—R*: a single non-reinforced reaction was sufficient to extinguish the conditioned reflex in this automaton. However, a single reinforcement is also sufficient to teach this reflex to it again: if we now serve simultaneously the impulses *UCS* and *CS* the memory element again transits to the inner state s_2 and learns the conditioned reflex.

Of course, the conditioned reflex in the automaton[13] of Fig. 19 is very simple. This automaton learns and forgets too quickly in order

13. This automaton was adapted from Henryk Greniewski, *Cybernetics without mathematics*. I have only added a self-consistent calendar. This required the introduction of the byrocrats.

to be realistic. It also needs both the conditioned stimulus *CS* and the unconditioned stimulus *UCS* simultaneously, and not successively as a real animal. It is not capable of generalizing the conditioned stimulus *CS* in the same way as a real animal is. However, these more realistic features could be easily reached by a more complicated finite automaton.

7 / Analog Systems

In a digital system the interaction between the system and the environment, and between the different parts of the system, occurs in the form of pulsating phenomena such as the beats of heart or the nervous impulses. Such pulsating phenomena are rather rare in the (macrophysical) material reality, at least outside living organisms. It seems as if nature had only with difficulty succeeded in producing digital systems. The most typical approaches to a digital system in nature are the nervous systems of men and the animals.

A typical macrophysical causal system is not a digital system: usually in non-living nature we cannot distinguish between different impulses of interaction, but interaction is more or less continuous in both space and time. Such cybernetic systems are called analog systems, for reasons that will be explained later.

A typical (causal) physical system, and thus an analog system, is, for instance, the system of gas molecules in a closed box which we have already discussed before. In such a system for the volume V of the gas, the gas pressure p, and the temperature T (measured as absolute temperature in the Kelvin scale) we have the equation

$$\frac{pV}{T} = c = \text{const.}$$

Thus, whatever changes p, V, and T may go through in the course of time the product pVT^{-1} is always constant (in an ideal gas).

Accordingly, if we press (cf. Fig. 20) the box, or the cylinder in which the gas is closed, so that the volume is decreased by ΔV, then the gas pressure p and the gas temperature T are increased by the respective amounts Δp and ΔT so that the changed values again obey the above law:

$$\frac{(p+\Delta p)(V+\Delta V)}{T+\Delta T} = \frac{pV}{T} = c.$$

By expanding the product we get $(p+\Delta p)(V+\Delta V) = pV+p\Delta V+V\Delta p+$ $+\ \Delta p\Delta V$. But if ΔV, Δp, and ΔT are small we can eliminate here $\Delta p\Delta V$

as a second order small magnitude, and get $(p+\Delta p)(V+\Delta V) = pV+ +p\Delta V+V\Delta p = pV+\Delta(pV)$. Here we have used the rule of derivation of a product according to which $\Delta(pV) = p\Delta V+V\Delta p$ for small Δp and ΔV. When substituting the result for the above formula we get

$$\frac{pV+\Delta(pV)}{T+\Delta T} = \frac{pV}{T} = c.$$

Accordingly the changes Δp, ΔV, and ΔT obey the law

$$p\Delta V+V\Delta p = c\Delta T.$$

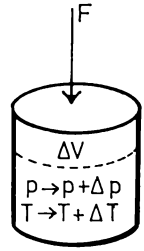


Fig. 20. An analog system where time is not explicitly introduced: the gas in a cylinder

Now it is evident that when we press the cylinder and thus change the volume of the gas by ΔV, the pressure and the temperature are not suddenly changed by the respective amounts Δp and ΔT determined by the above law. The pressure and the temperature are rather increased continuously to the values of *equilibrium* indicated by the law. In fact the increase of the pressure p and the temperature T as a consequence of the decrease of the volume V are time processes but our consideration of the gas system has not explicitly involved time. Still the system obviously is causal. *This is an example which shows that an explicit introduction of time is by no means necessary in the study of causality*, and thus of cybernetic system. *If time is not explicitly introduced, we only have, instead of the laws of time processes, the laws which characterize the states of equilibrium to which such processes tend.*

Let us now consider a typical physical analog system where time is explicitly introduced, and thus the time processes can be explicitly described. Let us consider a mass point having the mass m, and moving along the x-axis. The position of the mass point at time t is then indicated

by its x-coordinate at time t: $x(t)$. The velocity of the mass point is represented by the time-derivative $\dot{x}(t) = \frac{dx}{dt}(t)$, and the acceleration by the second derivate $\ddot{x}(t) = \frac{d^2x}{dt^2}(t)$. Let a time-dependent force $u(t)$ act on the mass point. Then the dynamical law which determines its motion can be written

$$m\,\ddot{x}(t) = u(t).$$

By integrating from t_0 to t we get on the left side:

$$m\int_{t_0}^{t}\ddot{x}(t)dt = m\int_{t_0}^{t}\frac{d\dot{x}}{dt}(t)\,dt = m\int_{t_0}^{t}d\dot{x} = m\,(\dot{x}(t) - \dot{x}(t_0)),$$

and on the right side:

$$\int_{t_0}^{t}u(\tau)\,d\tau.$$

Thus for the equations of motion we get the first integrated form

$$\dot{x}(t) = \dot{x}(t_0) + \frac{1}{m}\int_{t_0}^{t}u(\tau)\,d\tau$$

Another integration then yields the final integrated form

$$x(t) = x_0 + (t - t_0)\,\dot{x}_0 + \frac{1}{m}\int_{t_0}^{t}d\sigma\int_{t_0}^{\sigma}u(\tau)\,d\tau$$

Here we have denoted $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$.

The last equation expresses the position $x(t)$ of the particle at time t in terms of the combination $(x_0, \dot{x}_0)$ referring to the time point t_0, in terms of the values of the force function u in the interval $K_{t_0}^{t} = (t_0, t)$ and in terms of the end points t and t_0 of this interval. Accordingly, it can be regarded to define the output function of a non-time-invariant state-determined system. In this output function we can represent the inner state of the system as a 2-component vector

$$s(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}.$$

Then the output function can be written

$$x(t) = (1 \;\; t - t_0)\cdot s(t_0) + \frac{1}{m}I(u) = f(s(t_0),\; u|K_{t_0}^{t}),$$

where the functional $I(u)$ is

$$I(u) = \int_{t_0}^{t}d\sigma\int_{t_0}^{\sigma}u(\tau)\,d\tau$$

The state-transition function g is represented in a similar way by a combination of the two integrated forms of the equations of motion. The state-transition can be written:

$$s(t) = \begin{pmatrix} 1 & t-t_0 \\ 0 & 1 \end{pmatrix} s(t_0) + \frac{1}{m} \begin{pmatrix} I(u) \\ \mathrm{d}I(u)/\mathrm{d}t \end{pmatrix} = g(s(t_0), u|K^t_{t_0}).$$

Here

$$\frac{\mathrm{d}I}{\mathrm{d}t}(u) = \int_{t_0}^{t} u(\tau)\,\mathrm{d}\tau.$$

If we choose t_0 to be smaller than t, then $s(t_0)$ represents the initial state of the system, and the input segment $u|K^t_{t_0}$ represents the force which acts on the system after the time t_0 up to the time t when the output $x(t)$ is given. Thus in this case the solution of the equations of motion for $x(t)$ or $s(t)$ is non-anticipatory: they connect the output $x(t)$ and the state $s(t)$ with a past input and a past inner state. In this case the output relation $x(t) = f(s(t_0), u|K^t_{t_0})$ and the state-transition relation $s(t) = g(s(t_0), u|K^t_{t_0})$ can thus be taken as a formal expression of causality.

However, we can just as well choose t_0 greater than t, and the above solutions of the equations of motion for $x(t)$ and $s(t)$ will still be formally valid. In this case the relations $x(t) = f(s(t_0), u|K^t_{t_0})$ and $s(t) = g(s(t_0), u|K^t_{t_0})$, are anticipatory: they connect the output $x(t)$ and the state $s(t)$ with a future input and a future inner state. Of course the latter way of representing the solution is not very useful since in practice we do not know the future values of physical magnitudes. Still such a representation is formally possible. This illustrates the fact that the formal description of the explicit time process, whether anticipatory or non-anticipatory, has nothing to do with the existence of causality. The existence of causality is independent of the way in which the time process is formally described. The existence of causality depends only on the true sequence of real events in each case, and it can be only analyzed on an informal basis.

If we know the output function f and the inner state of a cybernetic system, we can use this system as a computer which computes the values of f (the output) for given inputs. Of course we have to construct the system so that its output function does the particular calculation which we want to be done. In this way we can replace the arithmetical process of calculation by a physical process, or represent the process of cal-

culation by a physical analogy. Hence the name analog system for a non-digital cybernetic system.

The digital systems can of course also be used as computers. Indeed they are the prototypes of modern electronic computers. However, in a digital system the computation is not realized by replacing an arithmetical process by a physical analogy. In digital computers the arithmetical process is preserved: the binarized inputs and outputs are interpretable as digits themselves. Hence the name digital.

Of course the examples of cybernetic systems, both digital and analog, represented above were rather trivial. Their purpose was only to give simple illustration of the foregoing definitions.

3 § The Notion of Cybernetic Whole

1 / Wholes and Components

Let us consider N cybernetic systems $S_1, \ldots, S_N$. Each of them, say S_r, is a relatively isolated material object, M_b^r, whose interaction with the environment is characterized by a causal relation R_c^r between a set X_r of input states and a set Y_r of output states:

$$S_r = (M_b^r, R_c^r) \quad \text{where} \quad R_c^r = \mathrm{Ch}\, M_b^r \subset X_r \times Y_r .$$

The set X_r is the cartesian product of some input or cause objects $X_1^r, \ldots, X_{m_r}^r$, and Y_r is the cartesian product of some output or effect objects $Y_1^r \ldots, Y_{n_r}^r$:

$$X_r = X_1^r \times \ldots \times X_{m_r}^r ,$$
$$Y_r = Y_1^r \times \ldots \times Y_{n_r}^r .$$

Accordingly, each input state $x_r \in X_r$ of the system S_r is a sequence of the elements of input objects, and each output state $y_r \in Y_r$ of the system S_r is a sequence of the elements of output objects:

$$x_r = (x_1^r, \ldots, x_{m_r}^r) \in X_r ,$$
$$y_r = (y_1^r, \ldots, y_{n_r}^r) \in Y_r .$$

If the system's S_s being in a given output state y_s may somehow influence the system's S_r being in a given input state x_r, or if the system's

S_r being in a given output state y_r may influence the system's S_s being in a given input state x_s, we say that the systems S_s and S_r are coupled with one another. If S_s is coupled with S_r, and S_r with S_u, we say that S_s is indirectly coupled with S_u. If each of the systems $S_1, \ldots, S_N$ is either coupled or indirectly coupled with each other, these systems together form a cybernetic whole S. Accordingly,

we define a 'cybernetic whole' as a material object composed of directly or indirectly mutually coupled cybernetic systems.

Such a structured whole is not necessarily itself a cybernetic system. However, it is always a system in the sense of general systems theory (cf. p. 87).

Indeed, each of the subsystems, or *components*, say S_r, is associated with $m_r + n_r$ system objects, viz. $X^r_1, \ldots, X^r_{m_r}, Y^r_1, \ldots, Y^r_{n_r}$. Thus we can associate with the whole S a totality of $m_1+n_1+m_2+n_2+\ldots+ + m_N + n_N$ system objects, viz.

$$X^1_1, \ldots, X^1_{m_1}, \ldots, X^N_1, \ldots, X^N_{m_N}, Y^1_1, \ldots, Y^1_{n_1}, \ldots, Y^N_1, \ldots, Y^N_{n_N}.$$

The mutual coupling of the components $S_1, \ldots, S_N$ already introduces a relation between all these system objects. Accordingly, we can consider the whole S, composed of the $S_1, \ldots, S_N$, as a general system.

2 / Organization

Let us now consider a whole S composed of the cybernetic systems $S_1, \ldots, S_N$. For this purpose we may introduce the sets X and Y defined by

$$X = X_1 \times \ldots \times X_N = X^1_1 \times \ldots \times X^1_{m_1} \times \ldots \times X^N_1 \times \ldots \times X^N_{m_N},$$
$$Y = Y_1 \times \ldots \times Y_N = Y^1_1 \times \ldots \times Y^1_{n_1} \times \ldots \times Y^N_1 \times \ldots \times Y^N_{n_N}.$$

Let us call the elements $x \in X$ the *total input states* of the whole S, and the elements $y \in Y$ the *total output states* of the whole S. (Notice that X and Y are not the sets of input and output states of S in the sense we have used these terms in the context of a cybernetic system S). Each total input state x and total output state y can thus be represented as sequences of the input or output states of the components $S_1, \ldots, S_N$ in the following way:

$$x = (x_1, \ldots, x_N) = (x^1_1, \ldots, x^1_{m_1}, \ldots, x^N_1, \ldots, x^N_{m_N}) \in X,$$
$$y = (y_1, \ldots, y_N) = (y^1_1, \ldots, y^1_{n_1}, \ldots, y^N_1, \ldots, y^N_{n_N}) \in Y.$$

In the present chapter we shall perform a preliminary study of structured wholes by considering only its organization and without introducing explicitly time. We shall distinguish between functional and structural organization. By functional organization we shall understand a particular kind of systematics appearing either in the total input x (*input organization*) or in the total output y (*output organization*). The study of input organization will give us the notion of information, and the study of output organization the notion of action. By *structural organization* we shall understand, first, the coupling of the individual outputs y_s to the individual inputs x_r and, secondly, the coupling by which the individual inputs x_r determine the individual outputs y_r. The term 'individual' was used here to indicate that we were speaking of a single component of S. (See the scheme on the following page).

In Chapter III we shall continue the study of cybernetic wholes. We shall study there in more detail the mode of action and the inner laws of motion of a cybernetic whole, introducing time explicitly.

3 / Structural Organization: Cybernetic Coupling

We want to specify mathematically what is meant by structural organization in a cybernetic whole S composed of the components $S_1, \ldots, S_N$. For this purpose we assume that each of the system objects

$$X^1_1, \ldots, X^1_{m_1}, Y^1_1, \ldots, Y^1_{n_1}, \ldots, X^N_1, \ldots, X^N_{m_N}, Y^N_1, \ldots, Y^N_{n_N}$$

of the general system S forms a *variable* of whose possible value is given by a numerical value times a unit which indicates the dimension (quality) of the variable in question. Thus we allow the possibility that different system objects represent different qualities, and have different dimensions and units. However, we assume the system objects to have been so composed that all the elements belonging to one and the same system object have the same quality and the same unit. Remembering that each of the system objects, X^r_k or Y^r_k, is associated with a certain dimension and unit, we can represent the elements $x^r_k \in X^r_k$ or $y^r_k \in Y^r_k$ of each system object by the respective numerical values, i.e. by real numbers.

Furthermore, we can make the convention that the numerical value $x^r_k = 0$ represents the shortage in the quality associated with the system object X^r_k, and $y^r_k = 0$ represents the shortage in the quality Y^r_k. If X^r_k (or Y^r_k) is a "qualitative", dichotomous variable, the presence of this quality may be indicated by the numerical value $x^r_k = 1$ (or

Organization
Functional organization
Output organization
Organized action
Input organization
Notion of information
Structural organization
Input-output coupling in a component
Coupling between the components

$y_k^r = 1$). If X_k^r (or Y_k^r) is a quantitative variable, we can represent by the numerical values $x_k^r > 0$ and $x_k^r < 0$ (or $y_k^r > 0$ and $y_k^r < 0$) the respective positive or negative magnitudes of this variable.

An input state x_r of the component S_r can then be mathematically represented by an m_r-component column vector, and an output state y_r by an n_r-component column vector:

$$(13) \qquad x_r = \begin{bmatrix} x_1^r \\ x_2^r \\ \cdot \\ \cdot \\ x_{m_r}^r \end{bmatrix}, \quad y_r = \begin{bmatrix} y_1^r \\ y_2^r \\ \cdot \\ \cdot \\ y_{n_r}^r \end{bmatrix}.$$

We can now specify mathematically the mutual coupling between the components $S_1, \ldots, S_N$ in the following way. For each output variable Y_k^s we introduce an output channel (Y_k^s) for the system S_s, and for each input variable X_j^r an input channel (X_j^r) for the system S_r (see Fig. 21). If the output y_k^s gives a contribution to the input x_j^r we say that there is a coupling form the output channel (Y_k^s) to the input channel (X_j^r). The contribution itself, which is of course a real number, we shall denote by

$$(14) \qquad x_j^{r|s} = c_{jk}^{rs} \cdot y_k^s.$$

The coefficient c_{jk}^{rs} is a real number which will be called the *coupling parameter.*

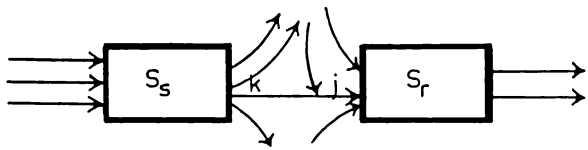


Fig. 21. The coupling from S_s to S_r

By different values of the coupling parameter we can indicate different kinds of coupling. These may be defined as follows:

$c_{jk}^{rs} > 0$: an excitative or positive coupling,

$c_{jk}^{rs} < 0$: an inhibitory or negative coupling,

$c_{jk}^{rs} = 0$: no coupling,

By the absolute value $|c_{jk}^{rs}|$ we can indicate the strength of the coupling in question:

$|c_{jk}^{rs}| > 1$: a strengthening coupling,

$|c_{jk}^{rs}| < 1$: a weakening or damping coupling,

$|c_{jk}^{rs}| = 1$: a standard coupling.

Since the value $c_{jk}^{rs} = 0$ expresses the case where is no coupling from the channel (Y_k^s) to the channel (X_j^r), we can apply the description of coupling by means of coupling parameters to every output channel of S_s and to every input channel of S_r. Thus the total coupling from S_s to S_r is indicated by the $m_r \times n_s$ matrix C_{rs} given by

$$C_{rs} = \begin{bmatrix} c_{11}^{rs} & c_{12}^{rs} & \dots & c_{1n_s}^{rs} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ c_{m_r1}^{rs} & c_{m_r2}^{rs} & \dots & c_{m_rn_s}^{rs} \end{bmatrix} .$$

This is the *coupling matrix* from the component S_s to the component S_r of the whole S.

The contribution $x_{r|s}$ of the total output y_s of S_s to the total input x_r of S_r can now be expressed as the matrix product

$$x_{r|s} = C_{rs} y_s . \tag{15}$$

By summing the contributions $x_{r|s}$ over the index s from 1 to N we get the *internal input* of the component S_r, i.e. the part of the input x_r which is due to the influence of the outputs of the other components of S:

$$x_r^{\text{int}} = \sum_{s=1}^{N} x_{r|s} = \sum_{s=1}^{N} C_{rs} y_s .$$

The input x_r may also contain contributions coming from the environment of the whole S. This part of x_r may be called the *external input* of the component S_r. Let us denote it by x_r^{ext}. It is of course also an m_r-component vector. Its components indicate the contributions of the environment of S to each of the m_r components of the input x_r. The input x_r is the sum of the internal and external inputs:

$$x_r = x_r^{\text{int}} + x_r^{\text{ext}} = \sum_{s=1}^{N} C_{rs} y_s + x_r^{\text{ext}} . \tag{16}$$

If $x_r^{\text{ext}} \neq 0$ the component S_r is a *receptor* of the whlole S. If $x_r^{\text{ext}} = 0$, S_r is a non-receptor component of S. We can also make a similar distinction with respect to the output y_r of the component S_r.

If the output y_r has a channel to the environment of the whole S, i.e. if y_r influences the environment of S directly, S_r may be called an *effector* of the whole S. On the other hand, if the output y_r influences directly only some other components of S, S_r is a non-effector component of S_r. If all the components $S_1, \ldots, S_N$ of the whole S are both non-receptors and non-effectors, the whole S is a *closed system*. As soon as the whole S has both receptors and effectors, it is of course a cybernetic system. If S has receptors but no effectors, or effectors but no receptors, we can call it *semi-closed*. Thus a cybernetic whole S may theoretically be of three main types:

A cybernetic whole is

a closed system	or	a semi-closed system	or	a cybernetic system

Let us return to the analysis of coupling. Let us arrange the individual coupling matrices C_{rs}, allowing both r and s to run from 1 to N, into a table so that the subscript r indicates the row and the subscript s the column in which the matrix C_{rs} is located. This arrangement gives us the *total coupling matrix* C of the whole S:

$$C = \begin{vmatrix} C_{11} & C_{12} \ldots . & C_{1N} \\ C_{21} & C_{22} \ldots . & C_{2N} \\ \ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots & & \\ C_{N1} & C_{N2} \ldots . & C_{NN} \end{vmatrix} = m \times n \text{ matrix.} \tag{17}$$

Since the individual coupling matrix C_{rs} has m_r rows and n_r columns, the total coupling matrix C has $m = m_1 + m_2 + \ldots + m_N$ rows and $n = n_1 + n_2 + \ldots + n_N$ columns. The total coupling matrix C gives a mathematical representation for the cybernetic *output-input couplings* between the components of a whole.

Let us then consider the *input-output coupling* within a given component, say S_r. Since S_r is a cybernetic system we know that it reacts to each input in a given input state x_r by an output in a given output state y_r. The output y_r depends on the input x_r which in general represents an input segment received by the system S_r. If we introduce time explicitly we can see that in the most general case each momentary output $y_r(t)$ of the cybernetic system S_r represents an accumulation

of the influences of a set of past momentary inputs. However, we shall not introduce time explicitly to our consideration of a structured whole yet — it will be done later on in Chapter III. Accordingly, we let here the input state x_r represent the time segment composed of all those momentary past inputs on which the output y_r depends. Then we indicate the existence of an input-output coupling in a cybernetic system S_r by writing

$$y_r = T_r\ (x_r). \tag{18}$$

Here T_r is an *action operator* which changes the input x_r received by the system S_r to the output y_r of the system. T_r may be either univalued (function) or many-valued: its mathematical structure does not interest us in this discussion, where we do not introduce explicitly time. (We shall return to this question in Chapter III.)

By arranging the individual action operators $T_1, \ldots, T_N$ into a matrix so that the operators fill the main diagonal while the other elements of the matrix are zero we get the *total action operator* T of the whole S:

$$T = \begin{vmatrix} T_1 & 0 & \ldots & 0 \\ 0 & T_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & T_N \end{vmatrix} . \tag{19}$$

This operator is a mathematical representation for the input-output couplings in all the components $S_1, \ldots, S_N$ of the whole S.

The combination (C, T) of the total coupling matrix C and the total action operator T is a mathematical representation for the *structural organization* or *cybernetic coupling* in a whole S.

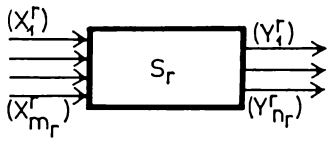


Fig. 22. Qualities represented by channels.

4 / Qualitative and Quantitative Aspects of System Objects

For the study of functional organization we must develop the formalism of a cybernetic whole S a bit further: we must consider the qualitative aspects of the system objects in more detail.

We will remember that a structured whole S is always a general (material) system. We shall use the same formalization of the system objects here as in the preceding section. Accordingly, we let each of the system objects

$$X_1^1, \ldots, X_{m_1}^1, \ldots, X_1^N, \ldots, X_{m_N}^N, Y_1^1, \ldots, Y_{n_1}^1, \ldots, Y_1^N, \ldots, Y_{n_N}^N$$

of the total system S represent a variable. The elements of the set X_j^r or Y_j^r are represented by the possible values of the respective variable X_j^r or Y_j^r. Each possible value of the variable X_j^r or Y_j^r is composed of a numerical part, the real number x_j^r or y_j^r, respectively, and of a unit indicating the quality or the dimension of the variable.

To each of the system objects, X_j^r or Y_j^r, we thus associate a certain quality (X_j^r) or (Y_j^r). To each *input quality* (X_j^r) there corresponds in the graphical representation of the system S_r an 'input channel', and to each *output quality* (Y_j^r) there corresponds an 'output channel' (see Fig. 22).

The different elements of the system object X_j^r or Y_j^r are then distinguished from one another by means of the different values of the real number x_j^r or y_j^r which indicates the numerical value of the variable X_j^r or Y_j^r.

Accordingly, we have distinguished between

the system objects:

$$X_1^1, \ldots, X_{m_1}^1, \ldots, X_1^N, \ldots, X_{m_N}^N, Y_1^1, \ldots, Y_{n_1}^1, \ldots, Y_1^N, \ldots, Y_{n_N}^N,$$

the qualities: $(X_1^1), \ldots, (X_{m_1}^1), \ldots, (X_1^N), \ldots, (X_{n_N}^N), (Y_1^1), \ldots, (Y_{n_1}^1), \ldots, (Y_1^N), \ldots, (Y_{n_N}^N)$, and

the magnitudes:

$$x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^N, \ldots, x_{n_N}^N, y_1^1, \ldots, y_{n_1}^1, \ldots, y_1^N, \ldots, y_{n_N}^N.$$

The component S_r has m_r input qualities and n_r output qualities. Let us denote the sets of its input and output qualities by Q_r and Q_r', respectively, so that we can write:

$$Q_r = \{(X_1^r), \ldots, (X_{m_r}^r)\}, \quad m_r = \# Q_r,$$
$$Q_r' = \{(Y_1^r), \ldots, (Y_{n_r}^r)\}, \quad n_r = \# Q_r'.$$

We shall call any subset of Q_r a *combined input quality* of the component S_r, and any subset of Q_r' a *combined output quality* of this component. The combined input qualities of S_r are thus the elements

of the set $F(Q_r)$ of all the subsets of Q_r, and the combined output qualities are the elements of the set $F(Q'_r)$ of all the subsets of Q'_r. Thus there are $q_r = 2^{m_r}$ combined input qualities, and $q'_r = 2^{n_r}$ combined output qualities in the component S_r:

$$q_r = \# F(Q_r) = 2^{m_r},$$
$$q'_r = \# F(Q'_r) = 2^{n_r}.$$

The whole S has the input qualities and the output qualities of all of its components $S_1, \ldots, S_N$. Thus the sets of the input and the output qualities of the whole S are given by the unions

$$Q = \bigcup_{r=1}^{N} Q_r \quad \text{and} \quad Q' = \bigcup_{r=1}^{N} Q'_r,$$

respectively. The whole S has $m = m_1 + \ldots + m_N$ input qualities and $n = n_1 + \ldots + n_N$ output qualities:

$$m = \# Q = \# Q_1 + \ldots + \# Q_n = m_1 + \ldots + m_N,$$
$$n = \# Q' = \# Q'_1 + \ldots + \# Q'_N = n_1 + \ldots + n_N.$$

Thus the number of the input (or output) qualities of the whole S is simply the *sum* of the input (or output) qualities of its components. This is trivial, if we remember that to each input (output) quality there corresponds an input (output) channel in the graphical representation: the number of all the input (output) channels within the whole S is of course the sum of the input (output) channels coming to its components.

The sum rule does not hold for the combined qualities of the whole S. The combined input qualities of the whole S are the subsets of Q, and its combined output qualities are the subsets of Q'. Thus their numbers are given by

$$q = \# F(Q) = 2^m = 2^{m_1 + \ldots + m_N} = q_1 q_2 \ldots q_N,$$
$$q' = \# F(Q') = 2^n = 2^{n_1 + \ldots + n_N} = q'_1 q'_2 \ldots q'_N.$$

Thus the number of the combined input (output) qualities of the whole S is the *product* of the numbers of the combined qualities of its components. This is easy to comprehend, since each combined input (output) quality corresponds to a set of input (output) channels.

Turning to the quantitative aspect of the system objects we have to consider the magnitudes $x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^N, \ldots, x_{m_N}^N, y_1^1, \ldots, y_{n_1}^1, \ldots, y_1^N, \ldots, y_{n_N}^N$, each of which is a real number. In the preceding section we already combined the input magnitudes $x_1^r, \ldots, x_{n_r}^r$ of the comp-

onent S_r to an input state vector x_r, and the output magnitudes $y_1^r, \ldots, y_{n_r}^r$ to an output state vector y_r. Now we can continue this construction by representing each possible total input state x of the whole S by a vector composed of the vectors $x_1, \ldots, x_N$. In a similar way we can let the total output states y of S be represented by the values of a vector composed of $y_1, \ldots, y_N$. So we get the *total input vector* x and the *total output vector* y of the whole S:

$$(20) \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_N \end{bmatrix} .$$

The set X_r of all the input states of the component S_r and the set Y_r of all the output states of S_r were given by

$$X_r = X_1^r \times \ldots \times X_{m_r}^r \text{ and}$$
$$Y_r = Y_1^r \times \ldots \times Y_{m_r}^r,$$

respectively. Now we shall make the assumption that each component system S_r has a finite number μ_r of possible input states, and a finite number ν_r of possible output states:

$$\mu_r = \# X_r = \# (X_1^r \times \ldots \times X_{m_r}^r) < \infty,$$
$$\nu_r = \# Y_r = \# (Y_1^r \times \ldots \times Y_{n_r}^r) < \infty.$$

The set X of all the total input states of the whole S and the set Y of all the total output states of S are given by

$$(21) \qquad \begin{aligned} X &= X_1 \times \ldots \times X_N, \quad \text{and} \\ Y &= Y_1 \times \ldots \times Y_N, \end{aligned}$$

respectively. Thus the numbers of input and output states of the whole S are obtained from the corresponding numbers of the component systems by the product rule:

$$\mu = \# X = (\# X_1) \ldots (\# X_N) = \mu_1 \mu_2 \ldots \mu_N,$$
$$\nu = \# Y = (\# Y_1) \ldots (\# Y_N) = \nu_1 \nu_2 \ldots \nu_N.$$

5 / Input Organization: The Notion of Input Information

We are now ready to discuss the first aspect of functional organization in a cybernetic whole, i.e. the organization of total input. We shall define the cybernetic notion of *input information* in terms of the properties of total input of a cybernetic whole.

Let us consider the total input vector

$$(22) \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} x_1^1 \\ \cdot \\ \cdot \\ x_{m_1}^1 \\ x_1^2 \\ \cdot \\ \cdot \\ x_{m_2}^2 \\ \cdot \\ \cdot \\ x_1^N \\ \cdot \\ \cdot \\ x_{n_N}^N \end{bmatrix}, \quad \text{where} \quad x_r = \begin{bmatrix} x_1^r \\ x_2^r \\ \cdot \\ \cdot \\ x_{m_r}^r \end{bmatrix}.$$

When a structured whole S is in a given total input state $x \neq 0$ we say that it receives some 'information' from this total input state. We shall first distinguish between the *content* and the *magnitude* of this notion called 'information'.

When speaking of the information content of a given total input state x we distinguish between information quality and information pattern. Let us first define the information quality Q_x of the total input state x.

If the component x_j^r of the vector x is different from zero,

$$x_j^r \neq 0,$$

we say that the whole S "receives" the input quality (X_j^r) when the whole is in the total input state x. We define the *information quality* Q_x of the total input state x as the set of all the input qualities the whole S receives when it is in the state x:

$$Q_x = \{(X_j^r);\ x_j^r \neq 0\}.$$

In other words, the information quality Q_x of the state x is the combined input quality which the whole S receives when it is in the state x. The set Q_x is of course a subset of the set Q of all the input qualities of the whole S, and an element of the set $F(Q)$ of all the combined input qualities of S:

$$Q_x \subset Q, \quad Q_x \in F(Q).$$

The set $F(Q)$ has $q = 2^m$ elements. Thus there are q possible information qualities which the whole S can receive from a total input state. We say this: the whole S is able to distinguish between q different information qualities.

Each of the component systems, say S_r, has only $q_r = 2^{m_r}$ combined input qualities, and thus is able to distinguish only between q_r different information qualities. The number q of the information qualities which the whole S can distinguish is the product of the information qualities which can be distinguished by the components: $q = q_1 q_2 \ldots q_N$. This is easy to comprehend, since an information quality corresponds to the set of those input channels along which a non-vanishing input is coming when the system is in a given input state.

The information quality Q_x is an aspect of the information content of a total input state x. Another aspect of its information content is the information pattern P_x of the state x. By the *information pattern* P_x we understand the distribution of the magnitudes of the non-vanishing components of x over the input qualities received by the system S when it is in the state x. Accordingly, we can define P_x as the restriction of the vector x to the vector components whose qualities are represented in the set Q_x:

$$P_x = x|Q_x.$$

Let us then consider the quantitative aspects of the information received by the whole S from a total input state x. Before the definition of the actual magnitude I_x of information contained in a given state x we shall study the information capacity of the system S.

Each of the total input states x is a choice among μ possible total input states (see the preceding section). We say that the *information capacity* I of the total system S is greater the greater the number μ of its possible total input states. To determine the mathematical relation between I and μ we ask for the number of successive yes-or-no choices we have to make in order to choose one among μ possibilities. If I has to be this number we have:

$$2^I = \mu.$$

Hence:

(23) $$I = \log_2 \mu$$

One counts information in terms of yes-or-no choices mainly because it is the most natural method of counting in the context of the most important systems of nature, the digital systems.

Each of the component systems, S_r, has μ_r possible input states. Thus the information capacity I_r of the component S_r is given by

$$I_r = \log_2 \mu_r .$$

Since $\mu = \mu_1 \mu_2 \ldots \mu_N$, the information capacity of the total system S is the sum of the information capacities of the components:

$$I = \log_2 \mu_1 + \ldots + \log_2 \mu_N = I_1 + \ldots + I_N .$$

The definition of information capacity takes into account only the number of the possible total input states from which a given total input state x is a choice. It does not take into account any previous experience the system may have. To define the *information magnitude* I_x received by the system S when it is in a given total input state x we have to take into account somehow the earlier experience that the system S has about the distribution of the total input states. This is because if, for instance, the system S has been all the past time in one and the same total input state x, the mere continuation of the state x does not give the system anything we want to call 'information'.

We want to define the magnitude I_x of information contained in the state x in terms of the 'unexpectedness' of the state x. To do this we can express the earlier experience of the system S of the distribution of the total input states by associating a probability $p(x)$ to each possible total input state x. The magnitude of information, I_x, has to be so defined that it is greater the greater the magnitude $1/p(x)$. In accordance with the above choice of unit we put:

(24) $$I_x = \log_2 \frac{1}{p(x)} .$$

If we consider the total system S during an interval of time which is long enough to bring forth the statistical frequencies $f(x) \approx p(x)$ of the total input states x, we get, for the *average information* I_{av} received by the system S during this interval, the formula

(25) $$I_{av} = \sum_{x \in X} p(x) I_x = \sum_{x \in X} p(x) \log_2 \frac{1}{p(x)} .$$

I_{av} is non-negative and has its maximum value when all the probabilities are equal, i.e. when $p(x) = 1/\mu$ for all $x \in X$. This maximum value is I:

$$0 \leq I_{av} \leq I = \log_2 \mu.$$

This explains the term "information capacity" we used above for the magnitude I. The average information I_{av} is sometimes called *information entropy* (sometimes this entropy is defined with the opposite sign).

Each of the component systems, S_r, receives from a given total input state x the magnitude $I_x(r)$ of information given by

$$I_x(r) = \log_2 \frac{1}{p(x_r)}.$$

Here $p(x_r)$ is the probability of the input state x_r of the component S_r. For the average information $I_{av}(r)$ received by the component system S_r we thus get:

$$I_{av}(r) = \sum_{x_r \in X_r} p(x_r)\ I_x(r) = \sum_{x_r \in X_r} p(x_r)\ \log_2 \frac{1}{p(x_r)}.$$

This satisfies also the respective condition with respect to the information capacity:

$$0 \leq I_{av}(r) \leq I_r = \log_2 \mu_r.$$

Let us study the relation between information magnitudes received by the total system S and by its components. The probability $p(x)$ is of course the probability of the co-occurrence of all the component input states $x_1, x_2, \ldots, x_N$ which together form the vector x:

$$p(x) = p(x_1 x_2 \ldots x_N).$$

We define the conditioned probability $p(x_k | x_1 x_2 \ldots x_{k-1})$ by

$$p(x_k | x_1 x_2 \ldots x_{k-1}) = \frac{p(x_1 x_2 \ldots x_k)}{p(x_1 x_2 \ldots x_{k-1})}.$$

Then we can expand the probability $p(x)$ as follows:

$$p(x) = p(x_1)\ p(x_2|x_1)\ p(x_3|x_1 x_2) \ldots p(x_N|x_1 x_2 \ldots x_{N-1}).$$

Accordingly, we get

$$(26) \qquad \begin{cases} I_x = \log_2 \frac{1}{p(x)} = I_x(1) + I_x(2|1) + I_x(3|12) + \\ + \ldots + I_x(N|12\ldots(N-1)) \\ I_{av} = I_{av}(1) + I_{av}(2|1) + \ldots + I_{av}(N|12\ldots(N-1)), \end{cases}$$

where we have used the notations

$$\begin{cases} I_x(k|12..(k-1)) = \log_2 \dfrac{1}{p\,(x_k|x_1x_2..x_{k-1})}\,, \\ I_{av}(k|12..(k-1)) = \sum\limits_{x\in X} p\,(x_1..x_N) \log_2 \dfrac{1}{p\,(x_k|x_1..x_{k-1})}\,. \end{cases}$$

Since by definition

$$p\,(x_k|x_1..x_{k-1}) \geq p\,(x_1..x_k),$$

we get the results

$$(27) \qquad \begin{cases} I_x \leq I_x(1) + .. + I_x(N), \\ I_{av} \leq I_{av}(1) + .. + I_{av}(N). \end{cases}$$

Accordingly, the information I_x and the average information I_{av} obtained by the total system S are both smaller than the sum of the respective information obtained by its components $S_1, .., S_N$.

This fact can be given the following verbal formulation. The *functional organization* of the total input x is expressed by the probabilities $p\,(x)$ of the states $x \in X$. Due to this functional organization the experience of the total system S of the distribution of the total input states x is greater than the experiences of a mere disorganized (uncoupled) collection of its components $S_1, .., S_N$: the experience of the total system contains even the knowledge on the probabilities of the co-occurrences, $p\,(x_1x_2..x_k)$ (from $k = 2$ to N), in addition to the individual probablities $p\,(x_1), .., p\,(x_N)$. Since the experience of the total system S as a structured whole is greater, the same total input state x contains less news for it than it contains for a mere disorganized collection of the components $S_1, .., S_N$. Indeed, the more functional organization (coupling) there is between the components of the total input state x, the farther is the probability $p(x)$ from the situation

$$p(x) = p(x_1)p(x_2)..p(x_N)$$

which holds good for a disorganized (uncoupled) collection of the components $S_1, .., S_N$. Only in the situation of disorganization we have the sum rule:

$$\begin{cases} I_x = I_x\,(1) + .. + I_x(N), \\ I_{av} = I_{av}\,(1) + .. + I_{av}(N). \end{cases}$$

To sum up what we have so far said on the notion of *input information* we can construct a table which indicates that we have distinguished

between content and magnitude, and between certain content notions and certain magnitude notions:

Input information:				
Content notions:			Magnitude notions:	
Information quality Q_x	Information pattern P_x	Information capacity I	Information magnitude I_x	Average information (Information entropy) I_{av}

We have defined these notions for the whole system S and for the components $S_1, \ldots, S_N$, and we have studied the mutual relations of the notions referring to the whole with respect to the notions referring to the components. This is all that we have so far done.

If we like we can decompose all our notions of information into respective internal and external information. This corresponds to the decomposition of the input vector x_r of each component S_r into the internal and external inputs,

$$x_r = x_r^{\text{int}} + x_r^{\text{ext}},$$

which was performed on p. 119. The internal input x_r^{int} comes to the component system S_r from the other components of the whole S, while the external input x_r^{ext} comes from the outer environment of the whole S. We can extend the decomposition to the total input vector by writing

$$x = x^{\text{int}} + x^{\text{ext}},$$

where

$$x^{\text{int}} = \begin{bmatrix} x_1^{\text{int}} \\ \cdot \\ \cdot \\ x_N^{\text{int}} \end{bmatrix} \quad \text{and} \quad x^{\text{ext}} = \begin{bmatrix} x_1^{\text{ext}} \\ \cdot \\ \cdot \\ x_N^{\text{ext}} \end{bmatrix}.$$

We can then distinguish the *internal informations* Q_x^{int}, P_x^{int}, I_x^{int}, $I_{\text{av}}^{\text{int}}$, received by the total system S from the internal input x^{int}, and coming from the components of S, from the respective *external information* Q_x^{ext}, P_x^{ext}, I_x^{ext}, $I_{\text{av}}^{\text{ext}}$ which it receives from the external

input x^{ext}, coming from the outer environment of the whole S. In a similar way we can distinguish between the internal and external information received by the component S_r. A closed system S of course receives only internal information.

If our cybernetic whole S is a finite physical object composed of molecules we can derive an interesting connection between its information capacity (information entropy) and its thermodynamic entropy. It follows from the statistical theory of thermodynamics that every finite material structure can have only a finite number of physical states which were called by Max Planck the 'elementary complexions' of that structure. The number μ_0 of the elementary complexions of the system S thus gives an upper limit for the number μ of the possible total input states of S:

$$\mu \leq \mu_0 .$$

For a material system S which has μ_0 elementary complexions the *thermodynamic entropy* $\mathfrak{S}$ is defined by

$$\mathfrak{S} = k \log_e \mu_0 .$$

The numerical value of the Boltzmann constant k is

$$k = 1.38\ 10^{-16} \frac{\text{erg}}{1°\text{C}},$$

where erg is an energy unit, and 1°C means one degree of temperature (Centigrade).

We can compare the formula of thermodynamic entropy with the formula we get for the *supremum information capacity* sup I of the system S, if we assume that every physical state of S is able to function as a total input state of S:

$$\sup I = \log_2 \mu_0 = \frac{1}{\log_e 2} \cdot \log_e \mu_0$$

We can, if we like, identify the supremum information capacity of the material system S with its thermodynamic entropy. This gives a ratio for the units of measurement used when measuring information and when measuring thermodynamic entropy. If we call the unit of information used above a "bit" (*binary unit*), we get:

$$1 \text{ bit} = \frac{\mathfrak{S}}{\sup I} = k \log_e 2 \approx 0{,}7\ k \approx 10^{-16} \frac{\text{erg}}{1°\text{C}} .$$

In this way we get for a bit of information as a physical equivalent a very little amount of energy per unit temperature.

We shall consider the similarity of meaning between thermodynamic entropy and cybernetic organization in connection with output organization and organized action.

We have in this section discussed the notion of information by defining it as a property of the total input states of a material system. From a purely formal point of view the same formalism could be applied to any other 'source of information' and not only to the total input states of a material system. Such an application is given, for instance, by the theory of information of Shannon and Weaver. Of course one can make useful application of the notion of information in the context of many kinds of 'sources of information'.

6 / Output Organization: The Cybernetic Notion of Action

We shall now discuss the second aspect of functional organization in a cybernetic whole, viz. the organization of total output. We shall define the cybernetic notion of *organized action* in terms of the total output of a structured whole.

Our study of output organization will be short, because the formalism developed in the preceding section can also be directly applied to the study of output organization. We have only to consider, instead of the total input, the total output vector

$$(28) \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix} = \begin{bmatrix} y_1^1 \\ \cdot \\ \cdot \\ y_{n_1}^1 \\ y_1^2 \\ \cdot \\ \cdot \\ y_{n_2}^2 \\ \cdot \\ \cdot \\ y_1^N \\ \cdot \\ \cdot \\ y_{N_n}^n \end{bmatrix}, \text{ where } y_r = \begin{bmatrix} y_1^r \\ y_2^r \\ \cdot \\ \cdot \\ \cdot \\ y_{n_r}^r \end{bmatrix}.$$

When a structured whole S is in a given total output state $y \neq 0$ *we say that it is in the 'action'* y . We shall first distinguish between notions of *action content* and *action organization.*

By the action content of a given state y of total output we mean the combination of the action quality Q_y and action pattern P_y of this state. The *action quality* Q_y is the set of all the output qualities (Y_j') for which the magnitude y_j' is different from zero in the state y:

$$Q_y = \{(Y_j');\ y_j' \neq 0\}.$$

In other words, the action quality Q_y of the output state y is the combined output quality which characterizes this state. The set Q_y is a subset of the set Q' of all the output qualities of the whole S and an element of the set $F(Q')$ of all the combined output qualities of S:

$$Q_y \subset Q',\quad Q_y \in F(Q').$$

The set $F(Q')$ has $q' = 2^n$ elements. We say this in the following way: the material system S has at its disposal q' different action qualities.

Each of the component systems, S_r, has only $q_r' = 2^{n_r}$ combined output qualities, and thus action qualities. The number q' of the action qualities of the whole S is the product of the numbers of the action qualities of the components $S_1, \ldots, S_N$: $q' = q_1' q_2' \ldots q_N'$. The whole S has an action quality for every set of output channels composed of the output channels of its component systems.

The *action pattern* P_y we define as the distribution of the magnitudes of the non-vanishing components of y over the output qualities of the system S in the state y. We can indicate this definition by writing P_y as the restriction of y to the vector components whose qualities are represented in the set Q_y:

$$P_y = y \mid Q_y.$$

Turning then to the quantitative aspects of action we define first the action capacity of the system S. We remember that the number of all the possible total output states of the system S was $\nu: \nu = \#Y$. We say that the *action capacity* C of the system S is the greater the greater is the number ν of its possible total output states. We measure the action capacity C in terms of the number of the yes-or-no choices involved in the choice of a particular action y. Accordingly, we put

$$2^C = \nu$$

so that

(29) $$C = \log_2 \nu.$$

In a similar way we define the action capacity C_r of the component system S_r in terms of the number ν_r of its output states:

$$C_r = \log_2 \nu_r.$$

Since $\nu = \nu_1 \nu_2 \ldots \nu_N$, the action capacity of the total system S is the sum of the action capacities of the components:

$$C = \log_2 \nu_1 + \ldots + \log_2 \nu_N = C_1 + \ldots + C_N.$$

For a notion which corresponds to the notion of information magnitude I_x we have no use in the context of action. In its stead we go directly to the counterpart of $I_{a\nu}$ in output organization. We assume that the functioning of the whole S is so organized that each particular action y has a given probability $p(y)$. (Of course these probablilities, and also the probabilities $p(x)$ of the total input states, can change in the course of time — we do not discuss the time process until Chapter III.) We then define the *action entropy* E of the system S by the formula

$$E = \sum_{y \in Y} p(y) \log_2 \frac{1}{p(y)} . \tag{30}$$

Then entropy E is non-negative and has its maximum value when all the probabilities are equal, i.e. when $p(y) = 1/\nu$ for all $y \in Y$. This maximum value is the action capacity of the system:

$$0 \leq E \leq C = \log_2 \nu.$$

The action entropy E_r of the component system S_r is given by

$$E_r = \sum_{y_r \in Y_r} p(y_r) \log_2 \frac{1}{p(y_r)} ,$$

where $p(y_r)$ is the probability of the output state y_r of S_r. Its maximum value is the action capacity C_r of the component system S_r:

$$0 \leq E_r \leq C_r = \log_2 \nu_r.$$

The relation between E and $E_1, \ldots, E_N$ is obtained by applying the formalism on pp 128—129. The result is

$$\begin{aligned} &E = E_1 + E(2|1) + E(3|12) + \ldots + E(N|12\ldots(N-1)), \\ &E \leq E_1 + E_2 + E_3 + \ldots + E_N. \end{aligned} \tag{31}$$

Here we have denoted

$$E(k|12\ldots(k-1)) = \sum_{y \in Y} p(y_1 \ldots y_N) \log_2 \frac{1}{p(y_k|y_1 \ldots y_{k-1})} .$$

Let us consider the significance of the action entropy E. For this purpose, let us study the functioning of the system S during an interval of time which is long enough to bring the frequencies $f(y)$ of the output states y close to the probabilities $p(y)$: $f(y) \approx p(y)$ for each $y \in Y$. Then

the entropy E calculated on the basis of these probabilities characterizes the *degree of organization* versus pure chance appearing in the actions of the system S during this interval of time. This can be shown by the following consideration.

Let us consider a situation in which the entropy E has its maximum value C:

$$E = \sum_{y \in Y} p(y) \log_2 \frac{1}{p(y)} = E_{\max} = \log_2 \nu.$$

This corresponds to the case in which all the probabilities $p(y)$ are equal, i.e.

$$p(y) = 1/\nu \quad \text{for all} \quad y \in Y.$$

Accordingly, the appearance of a given action y seems to follow from pure chance: there are no regularities, no organization in the acting of the system S during the interval of time in question.

Let us consider the conditions under which the equality of all the probabilities $p(y)$ is realized. The equality of the probabilities $p(y)$ for all possible $y \in Y$ implies the equality of the probabilities $p(y_r)$ for all possible $y_r \in Y_r$ for every component system S_r. Thus the maximum value of the total entropy E pressupposes the maximum values of the individual entropies E_r of the components S_r:

$$E_r = \sum_{y_r \in Y_r} p(y_r) \log_2 \frac{1}{p(y_r)} = \max E_r = \log_2 \nu_r,$$

because

$$p(y_r) = 1/\nu_r \quad \text{for all} \quad y_r \in Y_r.$$

However, this condition is not a sufficient one for the maximality of the total entropy E. To reach the maximum value of E the components $S_1, \ldots, S_N$ must be uncoupled with one another so that the whole S completely disorganized. Under this condition we get

$$p(y) = p(y_1)\, p(y_2) \ldots p(y_N)$$

which gives the sum rule

$$E = E_1 + E_2 + \ldots + E_N.$$

When this is true we get

$$E = \max E_1 + \ldots + \max E_N = \log_2 \nu_1 + \ldots + \log_2 \nu_N =$$
$$= \log_2 \nu = E_{\max}.$$

Thus in order that the actions of the total system S were completely disorganized must 1) the actions of each component system S_r be disorganized, and 2) the total system S must be a disorganized whole of its components, so that the components $S_1, \ldots, S_N$ are uncoupled with one another. This is the situation characterized by the maximum value of the action entropy E. The same situation is characterized, of course, by the minimum value of the action negentropy $-E$.

The value of the action entropy may thus decrease, or the value of the action negentropy $-E$ increase in two ways:

(1) by increasing the organization of the actions of the component systems S_r, indicated by the increasing negentropies $-E_r$, or

(2) by increasing the organization of the total system S as a whole of its components $S_1, \ldots, S_N$ which is indicated by the increasing difference

$$\Delta E = E_1 + \ldots + E_N - E.$$

We can thus say that the value of the action negentropy $-E$ measures directly the degree of organization in the total action of the whole S: the value of $-E$ increases with the increasing organization of the total action. We can also say: the more organized the action of the whole S is, the greater is the value of $-E$.

The maximal organization of the action is reached when $E = -E = 0$. This is the maximum value of the action negentropy $-E$.

Example. To illustrate the entropy considerations let us calculate the action entropy of a whole S composed of two components S_1 and S_2. If both of them have three output states, we can write their individual entropies as

$$E_1 = -p_1\log_2 p_1 - p_2\log_2 p_2 - p_3\log_2 p_3,$$
$$E_2 = -s_1\log_2 s_1 - s_2\log_2 s_2 - s_3\log_2 s_3.$$

We choose for the probabilities p_j and s_j the following numerical values:

$$(p_1, p_2, p_3) = (1/2,\ 1/4,\ 1/4),$$
$$(s_1, s_2, s_3) = (1/4,\ 1/4,\ 1/2).$$

Thus the numerical values of the individual entropies are in this case equal:

$$E_1 = E_2 = (1/2)\log_2 2 + 2(1/4)\log_2 4 = 3/2.$$

Assuming first that S_1 and S_2 are uncoupled with one another the probabilities p_{jk} of the co-occurrence of the output states of S_1 and S_2 are given by $p_{jk} = p_j s_k$. This gives the numerical values

$$p_{jk} = (1/8,\ 1/8,\ 1/4,\ 1/16,\ 1/16,\ 1/8,\ 1/16,\ 1/16,\ 1/8)$$

Hence the action entropy of the total system S_0 where the components S_1 and S_2 are uncoupled with one another is

$$E_0 = -\sum_{j=1}^{3}\sum_{k=1}^{3} p_{jk} \log_2 p_{jk} = 3.$$

Thus we have indeed the sum rule in this case:

$$E_0 = E_1 + E_2.$$

Let us then introduce a coupling between S_1 and S_2. This is reflected in the probabilities p_{jk} as a probability distribution which deviates from the above one. For instance, we van choose

$$p_{jk} = (1/4,\ 1/4,\ 1/8,\ 1/8,\ 1/8,\ 1/32,\ 1/32,\ 1,32,\ 1/32).$$

This gives for the total entropy of the system S the numerical value

$$E = 2\tfrac{3}{4} < 3 = E_0.$$

The functional coupling between the components S_1 and S_2 thus decreased the total entropy, while the negentropy $-E$ increased from -3 to $-2\tfrac{3}{4}$.

To sum up what we have said so far about output organization we can construct the following table to indicate that we have distinguished between certain notions related with the content, on the one hand, and the organization, on the other, of action:

Action:			
Action content:		Action organization:	
Action quality Q_y	Action pattern P_y	Action capacity C	Action entropy E or action negentropy $-E$

We have defined these notions for the whole system S, and for the components $S_1, \ldots, S_N$. We have also studied the mutual relations between the notions referring to the whole and the notions referring to the components. This is all we have done so far.

We can if we like speak of *internal* and *external actions* y (and y_r) according to the fact whether the output y (or y_r) goes to some component(s) S or to the outer environment of S. However, the same output y (or y_r) may influence both some component(s) of S and the outer environment of S. Thus we cannot decompose the output y (or y_r) in a similar way as we decomposed the input x (and x_r) to a sum of internal and external parts. Only if we know, for instance, that a given action

y (or y_r) acts either only on some components of S or only on the outer environment of S, we can calculate with y^{int} or y^{ext} (and with y_r^{int} or y_r^{ext}). Otherwise a quantitative distinction between internal and external action is impossible.

For a molecular physical system we have again the number μ_0 of the elementary complexions of the system S. The number of the elementary complexions of the system S is as well the upper limit of the number ν of possible output states:

$$\nu \leq \mu_0.$$

By comparing the possible values of the action capacity C with the value of the thermodynamic entropy S of the system S we get a theorem concerning the supremum action capacity:

$$\sup C = \sup E_{max} = \mathfrak{S} = k \log_e \mu_0.$$

The action capacity, or the maximal action entropy E_{max} of the system S reaches this supremum value, if the system S is able to use each of its μ_0 physical states (elementary complexions) as a possible total output state y.

Just as the action entropy E (or rather the action negentropy $-E$) measures the degree of action organization of the system S as an organized whole, the thermodynamic entropy (or rather the negentropy $-\mathfrak{S}$) measures the degree of *physical organization* in S as a physical system. The number μ_0 of the elementary complexions depends on many variables characterizing the physical system S, e.g. its temperature, total energy, etc. Just like the action organization the physical organization too can change in the course of time. There is a general law, called the *Law of Entropy* (see p. 92), according to which $\mathfrak{S}$ increases in every closed physical system S. Accordingly, in a closed physical system organization decreases. In a cybernetic system — which by our definition is always open — the organization as measured by $-E$ may increase as a consequence of the interaction of the system with its environment.

7 / Levels of Organization

In the preceding sections we have considered a cybernetic whole S which was composed of N components $S_1, \ldots, S_N$. Usually, when studying a whole S, it is useful to decompose it to a few *first components*

$$S_1^{(1)}, \ldots, S_{N_1}^{(1)},$$

and then each of these, $S_i^{(1)}$, to a few *second components*

$$S_{i1}^{(2)}, \ldots, S_{iN_{i2}}^{(2)},$$

then each of these, $S_{ij}^{(2)}$, to *third components*

$$S_{ij1}^{(3)}, \ldots, S_{ijN_{ij3}}^{(3)},$$

etc. until the components $S_1, \ldots, S_N$ are reached. In this way the study of organization of S reduces to the study in different *levels of organization*. The first level of organization is formed by the coupling between the first components $S_{(1)}^1, \ldots, S_{N_1}^{(1)}$. The second level of organization comprehends the studies of the mutual couplings of the second components $S_{i1}^{(2)}, \ldots, S_{iN_{i2}}^{(2)}$ within each of the first components, and so on. Levels of organization are illustrated in Fig. 23.

At each level of organization we have to study the inner organization of a number of organized wholes, say $S_1', \ldots, S_K'$. The coupling matrix of one of them, S_k', is given by some matrix of the form

$$C_{kk}' = \begin{vmatrix} C_{11} & C_{12} & \ldots & C_{1M} \\ C_{21} & C_{22} & \ldots & C_{2M} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ C_{M1} & C_{M2} & \ldots & C_{MM} \end{vmatrix},$$

where M is the number of the components of S_k'. The coupling matrix of the *'higher order whole'* S' composed of the wholes $S_1', \ldots, S_K'$ will then be of the form

$$C' = \begin{vmatrix} C_{11}' & C_{12}' & \ldots & C_{1K}' \\ C_{21}' & C_{22}' & \ldots & C_{2K}' \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ C_{K1}' & C_{K2}' & \ldots & C_{KK}' \end{vmatrix}.$$

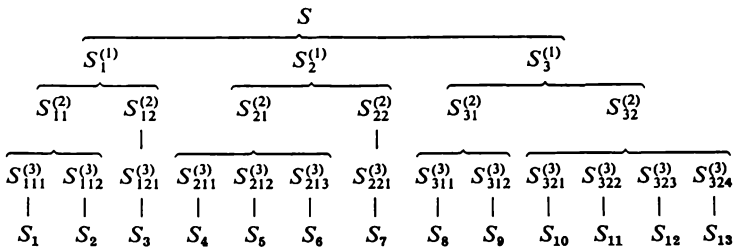


Fig. 23. Levels of organization of a whole S composed of 13 components $S_1, \ldots, S_{13}$.

Here the matrices C'_{jk} for $j \neq k$ contain the couplings between the wholes $S'_1, \ldots, S'_K$, while the matrices C'_{kk} contain the couplings within the respective wholes S'_k.

The action operator T', the total input vector x', and the total output vector y' of the higher order whole S' are simply the "direct sums" of the respective operators or vectors of the wholes $S'_1, \ldots, S'_K$:

$$T' = \begin{bmatrix} T_1 & 0 & \ldots & 0 \\ 0 & T_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & T_K \end{bmatrix}, \quad x' = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_K \end{bmatrix}, \quad y' = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_K \end{bmatrix}.$$

Thus we can study both the structural and functional organization separately at each level of organization.

CHAPTER III:

Cybernetic Theory of Self-Generating Processes

1 § The Self-Generating Process

1 / General

The world of dialectical materialism is a changing world. Dialectical materialism asserts that material reality is in a ceaseless state of change: it is self-generating, 'self-moving' for ever. The self-generated change, or 'motion', has a certain direction. Such a change is called *development.*

The world is pushed into motion by *dialectical contradictions* which are constantly formed and solved, and which generate themselves. The dialectical contradictions, which of course must be strictly distinguished from logical contradictions, are formed and solved as a consequence of causal interaction between the different parts of material reality[14]. Referring to the self-generating dialectical contradictions as the causal factor of development we can say that development is a *self-generating dialectical process.*

The dialectical contradictions are either *antagonistic* or *non-antagonistic* by their nature. The non-antagonistic contradictions can be removed and conciliated within the system where they are formed. The antagonistic contradictions break the system, and thus open a completely new line of development, which builds a new kind of system upon the ruins of the old one.

14. We are speaking here of dialectical processes ocurring in the objective, material reality. In addition to these processes dialectical materialism discusses processes of consciousness, which are reflections of the processes of material reality in human consciousness.

Corresponding to the two kinds of dialectical contradictions, non-antagonistic and antagonistic, a typical dialectical process is developed through a succession of two phases. First, there is a phase of gradual, accumulating development whose driving forces are both the antagonistic and the non-antagonistic contradictions within the system in question. During the period of gradual development the antagonistic contradictions, if such exist in the system in question, are sharpened until, as a consequence of their existence, an abrupt, qualitative change occurs in the system. In such a 'jump' the main qualitative characteristic of the system, viz. its *structure* and its *mode of action*, are abruptly changed. Hence, a new period of gradual development begins within the new system, leading to ever higher forms of organization.

A material system where a dialectical process of development occurs, as a consequence of the *internal contradictions* of this system, is often referred to as a *'whole'* in dialectical materialism. The internal contradictions of a whole are due to causal interactions between the parts of the whole in question. *External contradictions*, that is, contradictions due to causal interaction between the whole and its outer environment, may also contribute to the determination of the direction of the process. However, the internal contradictions, build within the structure and the mode of action of the whole, are the necessary moving forces which push the dialectical process into movement.

The whole of the universe can be considered as a whole where dialectical development occurs. However, the development of the universe as a whole is very slow. There are long periods of gradual development, in the course of which the mode of action of the universe, the natural laws, are changing very slowly.

Much more rapid is the dialectical process in the living population on the earth. The development of species shows periods of gradual change as well as abrupt changes. A striking qualitative change was the first appearance of human beings, capable of making tools and of cooperation, on the earth. Human societies, based on social relations formed by the cooperation and by the mutual communication of human beings by means of language, formed wholes which began dialectical processes themselves. The dialectical development pushed into motion by the internal and external contradictions of human societies is called the *historical process* of mankind.

In the course of the historical, dialectical process of development the structure and mode of action of the social formations of human beings undergo both gradual development and abrupt changes. The

former are called periods of *evolution*, the latter periods of *revolution*. Both evolution and revolution are necessary phases in the development of human societies.

We shall study in this chapter the formation of internal contradictions and dialectical process in a mathematical model. Such a model is offered by cybernetic method. However, we shall study here only a restricted aspect of the dialectical process: the structural and functional conditions under which a dialectical process is started in a whole with a given structure and a given mode of action. Thus we get a classification of internal contradictions, and a classification of the processes started by these contradictions. We shall not study the whole succession of the dialectical process composed of periods of gradual development and abrupt changes, leading to qualitatively higher forms of development. For a study of the latter problem the present cybernetic methods are insufficient. (A promising future approach to this problem seems to be in von Neumann's theory of the development of localized automata: see the remark on the role of complexity as a factor producing qualitative changes leading to higher forms of development, p. 169).

2 / Dialectical Contradiction Within a Cybernetic Whole

Let us consider a cybernetic whole S composed of the components $S_1, \ldots, S_N$. Our purpose is to study the internal conditions for the formation of dialectical process within this whole. To simplify the study we shall neglect irrelevant factors, and thus consider a whole S which is reduced to its 'active elements' coupled with one another. This means the following.

Each component S_r is a cybernetic system having m_r input channels and n_r output channels but only one inner state which never changes. Such a component is called an *active element*, since it does nothing but transforms a cause (the input) in a strictly deterministic, unique way to an effect (the output), thus changing its environment. The action operator T_r of an active element is thus a function, and associates with each input state x_r one and only one output state y_r:

$$y_r = T_r(x_r) \quad \text{(one-to-one)}. \tag{1}$$

Letting the channel along which an input comes determine the quality of the input in question, we can represent each input state x_r as an m_r-component real vector, whose component x_{rj} indicates the magnitude of the numerical value of the input coming along the j^{th} channel,

or the numerical index of the state component in question. Similarly, we represent each output state y_r by an n_r-component real vector (cf., p. 118).

In general, a given input state x_r can be expressed as

$$x_r = \sum_{s=1}^{N} C_{rs} y_s + x_r^{\text{ext}},$$

where C_{rs} is the $m_r \times n_s$ coupling matrix from S_s to S_r, and x_r^{ext} is the external input (cf. p. 119). However, we shall study the *internal process*, and put

$$x_r^{\text{ext}} = 0$$

for the whole time during we are observing the process (we assume that the system S has received earlier some external input, and we are now studying the internal process originated by it together with some internal background). Hence:

$$x_r = \sum_{s=1}^{N} C_{rs} y_s .$$

The total input vector x of the whole S has $m = m_1 + \ldots + m_N$ components, and the total output vector y has $n = n_1 + \ldots + n_N$ components. Written for them the coupling of the elements of S is expressed by

$$x = Cy. \tag{2}$$

Here C is the $m \times n$ matrix composed of the matrices C_{rs}, and called the total coupling matrix (cf. p. 120).

Each individual action operator T_r is a vector of n_r components. We can construct out of them the $n \times N$ quasi-diagonal matrix T, the total action operator of the whole S (cf. p. 121). The total action operator T indicates how a cause x acting upon the elements of the system S brings forth the effect y. On the other hand, the total coupling matrix C indicates how such an effect y acts as a new cause upon the elements. Thus T and C together represent the causal process occurring within the whole S.

Let us now pick out a certain total input state $\bar{x}$, and study its effects. We shall call $\bar{x}$ the *primary cause*. In the internal, causal process occurring in the system S the primary cause is first transformed to the primary effect $\bar{y}$:

$$\bar{x} \longrightarrow \bar{y} = T(\bar{x}). \tag{3}$$

Then the primary effect is transformed to the *first derived cause* $\bar{\bar{x}}$:

$$\bar{y} \rightarrow \bar{\bar{x}} = C\bar{y}. \tag{4}$$

Combining the two formulae, we have the *elementary causal act*

$$\bar{x} \rightarrow \bar{\bar{x}} = CT(\bar{x}) = R(\bar{x}). \tag{5}$$

We shall call $R = CT$ the *causal operator* of the whole S. It indicates how a primary cause is transformed into the first derived one in the internal, causal process of the whole S. The internal, causal process occurring in S can be thought of as a succession of such elementary causal acts.

If the first derived cause equals the primary cause, we say that the whole S is in a *state of equilibrium:*

$$\bar{\bar{x}} = \bar{x} \quad \text{(a state of equilibrium).}$$

If this is not the case, S is in a *state of change:*

$$\bar{\bar{x}} \neq \bar{x} \quad \text{(a state of change).}$$

If the whole S is in a *continuous* state of change, which is never stopped for cybernetic reasons (that is, neglecting the energy consumption of the process which may stop it), we say that there is an *internal dialectical contradiction* within the whole S. The non-stop internal, causal process which is so induced within the whole S is called a *self-generating process.* Thus a dialectical contradiction within a whole S always causes a self-generating process in this whole and, vice versa, a self-generating process is always due to an internal dialectical contradiction within the system in question.

The internal dialectical contradiction, if it exists, is said to be between the elements of the whole S in question. Or, more exactly, it is said to be between the inputs and the outputs of the individual elements of S.

To explain this terminology, let us consider a whole composed of only two elements S_1 and S_2. If there is an internal dialectical contradiction within the whole S, the two elements S_1 and S_2 can never be simultaneously in equilibrium. For, if S_1 is in equilibrium so that $\bar{\bar{x}}_1 = \bar{x}_1$, it follows from the condition $\bar{\bar{x}} \neq \bar{x}$ that there must be $\bar{\bar{x}}_2 \neq \bar{x}_2$. Thus the element S_2 'denies' the equilibrium of the element S_1 and, vice versa, S_2 if it is in equilibrium denies the equilibrium of S_1. This mutual denial represents a *dialectical contradiction.* If this term were not used, we would have to invent a new term expressing the same thing. So, for instance, the English cybernetician W. Ross Ashby introduced

the term 'power of veto' for dialectical contradiction as a driving force in the self-generating development of a cybernetic system[15]. However, in dialectical materialism no new term is needed.

The course of a self-generating process is determined by the causal operator R of the system or, in Marxist terminology, by the *mode of action* of the system. This in turn is determined by the matrices C and T, or by the *structure* of the system, while the action operator T indicates the *modes of action of the elements.*

Causality, expressed by the causal operator, or the mode of action R of the whole S, was defined above, just as it was defined originally for a material cybernetic system (cf. p. 94—96): without an explicit introduction of time. This is a characteristic trait of cybernetics as a method of dialectical materialism: in cybernetics, and in dialectical materialism in general, causality is the primary thing.

Note. Instead of using the causal operator R = CT we could as well have used the causal operator P = TC in our analysis of self-generating processes. Then we would only study the output process y(t) instead of the input process x(t). For convenience, we shall use P instead of R in our analysis of production process in Vol. II (because y has less components than x, and is thus easier to handle in computations).

3 / The Inner Law of Motion of a Whole

We shall now introduce time explicitly into the mode of action of a whole S. The new form of the mode of action so obtained is called the *Inner Law of Motion* of the whole. The term was introduced by Marx[16].

Let us consider the effect of a primary cause x appearing in the whole S at the moment t. If the primary effect y is a sudden act it appears at a definite moment $t+\theta$ of time not before the cause x: $\theta > 0$. We can in this case write the connection between cause effect as follows:

$$y(t+\theta) = T(x(t)), \quad \theta > 0 \quad \text{(sudden effect).} \tag{6}$$

If the effect of x is gradual, it is distributed over some interval $(t, t+\theta)$ of time, and we can write:

$$y(t+\tau) = T(x(t), \tau) \neq 0 \text{ for } 0 < \tau \leq \theta \quad \text{(gradual effect).} \tag{7}$$

15. W. Ross Ashby, *Introduction to Cybernetics*, London 1956, p. 100—101.
16. Karl Marx, *Capital*, Vol. I, Moscow 1954, p. 10.

To indicate the distribution of the effect y over the interval $(t,t+\theta)$ we have written the action operator T to depend explicitly on the time τ passed since the appearance of the cause x. We call θ the *reaction time* of the whole S. It will be assumed constant in our discussion.

Assuming that the primary effect $y(t+\theta)$, or $y(t+\tau)$, is immediately, at the moment $t+\theta$ or $t+\tau$, effective as a first derived cause $x(t+\theta)$, or $x(t+\tau)$, without any loss of time in the channels of coupling, we have:

$$\text{(I)} \quad \begin{cases} x(t+0) = CT(x(t)) = R(x(t)) & \text{(sudden effect),} \\ x(t+\tau) = CT(x(t),\tau) = R(x(t),\tau) \neq 0 \quad \text{for} \quad 0 < \tau \leq \theta & \text{(gradual effect).} \end{cases}$$

This is the first form of the Inner Law of Motion of the whole S. Of course we could have introduced some further parameters to indicate the times passed by the primary effects y_s in the channels of coupling before theys are transformed to the first derived causes x_r. This, however, would be an irrelevant complication from the point of view of our present study.

A second form of the Inner Law of Motion is obtained by writing the above formulae (I) for the first derived cause $x(t)$ instead of $x(t+\theta)$ or $x(t+\tau)$. Then we obtain:

$$\text{(II)} \quad \begin{cases} x(t) = CT(x(t-\theta)) = R(x(t-\theta)) & \text{(sudden effect),} \\ x(t) = C \sum_{\tau_1}^{\tau_n=\theta} T(x(t-\tau),\tau) = \sum_{\tau_1}^{\theta} R(x(t-\tau),\tau) & \text{(step-wise gradual effect),} \\ x(t) = C \int_0^{\theta} T(x(t-\tau),\tau)\,d\tau = \int_0^{\theta} R(x(t-\tau),\tau)d\tau & \text{(continuous gradual effect).} \end{cases}$$

We see that the Inner Law of Motion, whether given in the form (I) or (II) or for a sudden or a gradual effect, is a *recursive formula.* It represents the *causal recursion* by means of which the values of x for any moments $t \geq z$ of time can be calculated from the values of x in the interval $(z-\theta, z)$. The values in this interval can be called the initial values of the process. We can choose the moment z and the initial values of the process arbitrarily. The segment of the internal process $x(t)$ in any interval $(z-\theta, z)$ then determines uniquely the course of the internal process for all the future after the time z.

The causal operator $R(x)$ is an m-component vector function of the m-component vector variable x. If R is analytic in an environment of a point $\hat{x}$, we can expand the value $R(x)$ in the Taylor series (cf. p. 65)[17]:

$$(8) \qquad R(x) = R(\hat{x}) + \left(\frac{\partial R}{\partial x}\right)_{x=\hat{x}} (x-\hat{x}) + \frac{1}{2!}(x-\hat{x})' \left(\frac{\partial^2 R}{\partial x^2}\right)_{x=\hat{x}} (x-\hat{x}) + \ldots,$$

which converges, and represents the function $R(x)$ in some environment of the point $\hat{x}$. This gives the possibility of relating, if the necessary conditions of analyticity and convergence are fulfilled, any two solutions $x(t)$ and $\hat{x}(t)$ of the Inner Law of Motion, (II), by means of a power series in $x(t-\theta) - \hat{x}(t-\theta)$, or $x(t-\tau) - \hat{x}(t-\tau)$, respectively. We get for sudden effect:

$$(9) \qquad x(t) = \hat{x}(t) + \left[\frac{\partial R}{\partial x\,(t-\theta)}\right]_{x=\hat{x}} [x(t-\theta) - \hat{x}(t-\theta)] + \frac{1}{2!}[(x(t-\theta) - \hat{x}(t-\theta)]' \left[\frac{\partial^2 R}{\partial x^2 (t-\theta)}\right]_{x=\hat{x}} [x(t-0) - \hat{x}(t-0)] + \ldots$$

Similarly we get for step-wise gradual effect:

$$(10) \qquad x(t) = \hat{x}(t) + \sum_{\tau_1}^{\tau_n=0} \left[\frac{\partial R}{\partial x(t-\tau)}\right]_{x=\hat{x}} [x(t-\tau) - \hat{x}(t-\tau)] + \frac{1}{2!}[x(t-\tau) - \hat{x}(t-\tau)]' \sum_{\tau_1}^{0} \left[\frac{\partial^2 R}{\partial x^2 (t-\tau)}\right]_{x=\hat{x}} [x(t-\tau) - \hat{x}(t-\tau)] + \ldots$$

And for continuous gradual effect:

$$(11) \qquad x(t) = \hat{x}(t) + \int_0^0 \left[\frac{\partial R}{\partial x(t-\tau)}\right]_{x=\hat{x}} [x(t-\tau) - \hat{x}(t-\tau)]\,d\tau + \frac{1}{2!}\int_0^0 [x(t-\tau) - \hat{x}(t-\tau)]' \left[\frac{\partial^2 R}{\partial x^2 (t-\tau)}\right]_{x=\hat{x}} [x(t-\tau) - \hat{x}(t-\tau)]\,d\tau + \ldots$$

17. $\frac{\partial^2 R}{\partial x^2}$ is a tensor of third degree, as a "tree-dimensional matrix".

When written for the *difference process*

$$\Delta x(t) = x(t) - \hat{x}(t)$$

this reads:

$$(12)\begin{cases} \Delta x(t) = \left[\dfrac{\partial R}{\partial x(t-\theta)}\right]_{x=\hat{x}} \cdot \Delta x(t-\theta) + \dfrac{1}{2!}\,\Delta x'(t-\theta)\left[\dfrac{\partial^2 R}{\partial x^2(t-\theta)}\right]_{x=\hat{x}} \\ \qquad\qquad \Delta x(t-\theta) + \ldots \text{ (sudden effect)}, \\ \Delta x(t) = \sum_{\tau_1}^{0}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{x=\hat{x}} \cdot \Delta x(t-\tau) + \dfrac{1}{2!}\sum_{\tau_1}^{0}\Delta x'(t-\tau)\left[\dfrac{\partial^2 R}{\partial x^2(t-\tau)}\right] \\ \qquad\qquad \Delta x(t-\tau) + \ldots \text{ (step-wise gradual effect)}, \\ \Delta x(t) = \int_0^{\theta}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{x=\hat{x}} \cdot \Delta x(t-\tau)\,d\tau + \dfrac{1}{2!}\int_0^{\theta}\Delta x'\ (t-\tau) \\ \left[\dfrac{\partial^2 R}{\partial x^2(t-\tau)}\right]_{x=\hat{x}} \cdot \Delta x\,(t-\tau)\,d\tau + \ldots \text{ (continuous gradual effect)}. \end{cases}$$

In the domain where the Taylor series of R converges, and represents the function $R(x)$ one can calculate, by means of the equations (12), all the later values of the difference process Δx from its initial values in an arbitrarily chosen interval $(z-0,z)$.

In particular, for a *differential process*

$$\delta x(t) = x(t) - \hat{x}(t),$$

where $\delta x(t)$ is for all t so small that the magnitudes of the second order in $\delta x(t)$ can be ignored, we get:

$$(13)\begin{cases} \delta x(t) = \left[\dfrac{\partial R}{\partial x(t-0)}\right]_{x=\hat{x}} \cdot \delta x(t-0) \quad \text{(sudden effect)}, \\ \delta x(t) = \sum_{\tau_1}^{0}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{x=\hat{x}} \cdot \delta x(t-\tau) \quad \text{(step-wise gradual effect)}, \\ \delta x(t) = \int_0^{0}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{x=\hat{x}} \cdot \delta x(t-\tau)\,d\tau \quad \text{(continuous gradual effect)}. \end{cases}$$

Let us study the matrix $\partial R/\partial x$ appearing here. Since $R = CT$, and since C is composed of the coupling matrices C^{rs}, and since T is a

quasi-diagonal matrix composed of the individual action operators T_s we can represent R in following form:

$$R = \begin{bmatrix} R_{11} & R_{12} \ldots R_{1N} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ R_{N1} & R_{N2} \ldots R_{NN} \end{bmatrix} = \begin{bmatrix} C^{11} & C^{12} \ldots C^{1N} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ C^{N1} & C^{N2} \ldots C^{NN} \end{bmatrix} \cdot \begin{bmatrix} T_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & T_N \end{bmatrix}$$

Here each sub-matrix $R_{rs} = C^{rs}T_s$ has m_r rows and one column:

$$(14)\quad R_{rs} = \begin{bmatrix} C^{rs}_{11} & C^{rs}_{12} \ldots C^{sr}_{1n_s} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ C^{rs}_{m_r 1} & C^{rs}_{m_r 2} \ldots C^{rs}_{m_r n_s} \end{bmatrix} \cdot \begin{bmatrix} T_{s1} \\ T_{s2} \\ \vdots \\ T_{sn_s} \end{bmatrix} = \begin{bmatrix} (R_{rs})_1 \\ (R_{rs})_2 \\ \vdots \\ (R_{rs})_{m_r} \end{bmatrix}.$$

Each component T_{sk} of the action operator T_s of the element S_s is a function of all the components $x_{s1}, x_{s2}, \ldots, x_{sm_s}$ of the input vector x_s:

$$T_{sk} = T_{sk}(x_{s1}, x_{s2}, \ldots, x_{sm_s}).$$

Accordingly, each component $(R_{rs})_l$ of the sub-matrix R_{rs} has a derivative with respect to each component x_{sh} of x_s. This derivative can be written:

$$(15)\qquad \frac{\partial R_{rl}}{\partial x_{sh}} = \frac{\partial (R_{rs})_l}{\partial x_{sh}} = \sum_k c^{rs}_{lk} \frac{\partial T_{sk}}{\partial x_{sh}}$$

The derivative of the vector function $R(x)$ with respect to the vector x is the matrix composed of these derivatives:

$$(16)\qquad \frac{\partial R}{\partial x} = \left\| \frac{\partial R_{rl}}{\partial x_{sh}} \right\| = \left\| \left(\frac{\partial R}{\partial x} \right)_{rl,sh} \right\| = m \times m.$$

As indicated above, the rows of the matrix $\partial R/\partial x$ are shown by the double index (ri), and the columns by the double index (sh). The matrix $\partial R/\partial x$ has, obviously, $m = m_1 + m_2 + \ldots + m_N$ rows and equally many columns.

When substituting the matrix elements of $\partial R/\partial x$ to the formula (13) we get the corresponding equations in a component form:

$$(17)\quad \delta x_{rl}(t) = \sum_s \sum_h \left[\frac{\partial R_{rl}}{\partial x_{sh}(t-\theta)} \right]_{x=\mathfrak{x}} \cdot \delta x_{sh}(t-\theta) \quad \text{(sudden effect)},$$

etc.

2 § *Cybernetic Systematics of Self-Generating Processes*

1 / The Study of Internal Contradictions 'in the Small': Cybernetic Categories of Contradictions

Let $\hat{x}(t)$ be one of the solutions of the Inner Law of Motion, (I) or (II), of the whole S. It then represents one possible internal process occuring in this whole.

Denoting the interval $(z—\theta,z)$ in the calendar K of the whole S by $K^z_{z-\theta}$,

$$K^z_{z-\theta} = \{t\,;\, z-\theta \leq t < z\} \subset K,$$

the initial values of the process $\hat{x}(t)$, when taken in the interval $K^z_{z-\theta}$, are represented by the time segment

$$\hat{x}\,|\,K^z_{z-\theta} \quad \text{(the initial values of } \hat{x} \text{ for the moment } z).$$

The initial values $\hat{x}\,|\,K^z_{z-\theta}$ for the moment z then determine uniquely all the later development of the process $\hat{x}\,(t)$ for $t \geq z$.

If $x\,(t)$ is another solution of the Inner Law of Motion of S, the initial values $x|K^z_{z-\theta}$ for the moment z determine uniquely all the later course of the process $x(t)$ for $t \geq z$. In particular, the initial values $\hat{x}\,|\,K^z_{z-\theta}$ and $x\,|\,K^z_{z-\theta}$ together determine uniquely the course of the difference process $\Delta x(t) = x(t)—\hat{x}(t)$ for $t \geq z$.

If the necessary conditions of analyticity and convergence are fulfilled[18], the later values of the difference process $\Delta x(t)$ can be calculated by means of the power series in the equations (12), p. 150, when the initial values $\hat{x}\,|\,K^z_{z-\theta}$ and $x\,|\,K^z_{z-\theta}$ are given. If the difference $(x—\hat{x})\,|\,K^\theta_{z-\theta}$ of the initial values is small enough, the values of the difference process $\Delta x(t)$ can be calculated, for some sufficiently small interval $K^{z+\varepsilon}_z$ of time, from the formula

18. Let it be here emphasized, once and for all, that a corresponding discussion of the effects of a disturbance of the internal process of a whole could be performed without the assumptions of analyticity and convergence related to the causal operator $R(x)$. However, the discussion would be mathematically complicated and uneasy. Therefore, we restrict ourselves to a model where the causal operator $R(x)$ has the necessary qualifications to allow an analytic treatment.

$$(18)\begin{cases}\Delta x(t) = \left[\dfrac{\partial R}{\partial x(t-\theta)}\right]_{\substack{x=\hat{x}\\ t=z}} \Delta x(t-\theta) \text{ for } t \in K_z^{z+\varepsilon} & \text{(sudden effect),}\\ \Delta x(t) = \sum_{\tau_1}^{\theta}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{\substack{x=\hat{x}\\ t=z}} \Delta x(t-\tau) \text{ for } t \in K_z^{z+\varepsilon} & \text{(step-wise gradual effect),}\\ \Delta x(t) = \int_0^{\theta}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{\substack{x=\hat{x}\\ t=z}} \Delta x(t-\tau)\mathrm{d}\tau \text{ for } t \in K_z^{z+\varepsilon} & \text{(continuous gradual effect).}\end{cases}$$

Here the derivatives, as indicated, are *taken at the moment* z of time.

Let us now introduce a vector function f_z defined by

$$(19)\begin{cases} f_z(t) = \left[\dfrac{\partial R}{\partial x(t-\theta)}\right]_{\substack{x=\hat{x}\\ t=z}} f_z(t-\theta) \text{ for all } t \geq z & \text{(sudden effect),}\\ f_z(t) = \sum_{\tau_1}^{\theta}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{\substack{x=\hat{x}\\ t=z}} f_z(t-\tau) \text{ for all } t \geq z & \text{(step-wise gradual effect),}\\ f_z(t) = \int_0^{\theta}\left[\dfrac{\partial R}{\partial x(t-\tau)}\right]_{\substack{x=\hat{x}\\ t=z}} f_z(t-\tau)\mathrm{d}\tau \text{ for all } t \geq z & \text{(continuous gradual effect).}\end{cases}$$

We have

$$(20)\qquad f_z(t) = \Delta x(t) \text{ for } t \in K_z^{z+\varepsilon},$$

while for the times $t \geq z+\varepsilon$ the values of the function f_z may differ from the values of the difference process. We call f_z the *trend function* of the difference process $\Delta x(t)$ "in the small", that is, in a case where the initial values $(x-\hat{x})\big|K_{z-0}^{z}$ are small quantities of the first order so that their second powers can be ignored.

By means of the trend function f_z we can study the response of the system S to small disturbances of its internal process $\hat{x}(t)$. By a small disturbance of the process $\hat{x}(t)$ we mean a displacement of its initial values, $\hat{x}\big|K_{z-0}^{z} \to (\hat{x}+\Delta x)\big|K_{z-0}^{z}$, such that $\Delta x = x(t) - \hat{x}(t)$, where $x(t)$ is also a solution of the Inner Law of Motion, and the values $\Delta x\big|K_{z-\theta}^{z}$ are small quantities of the first order.

We shall now study the response of the whole S to such a small disturbance of its internal process $\hat{x}(t)$, performing the solution in more detail in the case of sudden effect.

1. Solution for Sudden Effect. For convenience, we shall write in short

$$\left[\frac{\partial R}{\partial x(t-\theta)}\right]_{\substack{x=\hat{x}\\ t=z}} = \left(\frac{\partial R}{\partial \hat{x}}\right)_z = m \times m \text{ matrix.}$$

The defining equation of the trend function then is for sudden effect:

$$f_z(t) = \left(\frac{\partial R}{\partial \hat{x}}\right)_z f_z(t-\theta). \tag{21}$$

The elements of the matrix $\partial R/\partial \hat{x}$ being here independent of the time t (though they depend on the moment z), this is an ordinary difference equation for an m-component vector function. The substitution $f_z(t) = w\lambda^{t/\theta}$, where w is a constant m-component vector and λ is a number, yields:

$$w\lambda^{t/\theta} = \left(\frac{\partial R}{\partial \hat{x}}\right)_z w\lambda^{\frac{t}{\theta}-1}$$

Here the factors $\lambda^{t/\theta}$ cancel each other, and we are left with the eigen value equation

$$\left(\frac{\partial R}{\partial \hat{x}}\right)_z w = \lambda w, \text{ or } \left[\left(\frac{\partial R}{\partial \hat{x}}\right)_z - \lambda \cdot I\right] w = 0. \tag{22}$$

Here I is the $m \times m$ unit matrix.

Solutions $w \neq 0$ exist only when the determinant of the characteristic matrix vanishes:

$$\left|\left(\frac{\partial R}{\partial \hat{x}}\right)_z - \lambda \cdot I\right| = 0. \tag{23}$$

This is the characteristic equation (cf. p. 52) for the matrix $\left(\frac{\partial R}{\partial \hat{x}}\right)_z$. The left-side member is a polynomial of the m^{th} degree in the unknown λ. Accordingly (cf. p. 79), we have m roots:

$$\lambda_1, \lambda_2, \ldots, \lambda_m.$$

Each of them, of course, is a function of the parameter z:

$$\lambda_j = \lambda_j(z); \; j = 1,2, \ldots, m.$$

For each non-multiple root λ_j we have exactly one solution (λ_j, w_j), apart from an arbitrary constant factor in w_j, of the eigen value equation. The eigen vector w_j of course depends on z too:

$$w_j = w_j(z); \; j = 1,2, \ldots, m.$$

We get, from each such solution (λ_j, w_j) of the eigen value equation, a uniquely determined solution $f_z(t)_j = w_j\lambda_j^{t/\theta}$ of the equation (21) (up to a constant factor). Assuming that there are no multiple roots

the general solution of the difference equation (21) thus is:

$$f_x(t) = \sum_{j=1}^{m} w_j \lambda_j^{t/\theta}.$$

Each root λ_j is either a real or a complex number, and thus can always be written (cf. the Euler formula, p. 64) as

$$\lambda_j = r_j e^{i\omega_j} = r_j(\cos\omega_j + i\sin\omega_j).$$

Here r_j is a non-negative real number, viz. the module of λ_j: $|\lambda_j| = r_j$. The real number ω_j is the argument of λ_j: arg $\lambda_j = \omega_j$.

The root λ_j is real for $\omega_j = 0, \pi, \ldots$ The eigen vector w_j is of course then real too. For $\omega_j \neq 0, n\pi$ we have a complex root λ_j. In this case, obviously, the eigen vector w_j is complex too. Let us write:

$$w_j = a_j + ib_j,$$

where a_j and b_j are real vectors.

For each complex root λ_j the conjugate complex number λ_j^* is also one of the m roots (cf. p. 79). Let us write $\lambda_j^* = \lambda_k$. The sum of the j^{th} and the k^{th} terms in the solution $f_x(t)$ can then be made real by choosing $w_k = w_j^*$, as can readily be seen from the condition that

$$(w_j\lambda_j^{t/\theta} + w_k\lambda_j^{*t/\theta})^* = w_j\lambda_j^{t/\theta} + w_k\lambda_j^{*t/\theta}.$$

Then we obtain:

$$w_j\lambda_j{}^{t/\theta} + w_k\lambda_k{}^{t/\theta} =$$
$$= 2a_j r_j^{t/\theta}\cos\omega_j t/\theta + 2b_j r_j^{t/\theta}\sin\omega_j t/\theta = \text{real}.$$

In this way we get a real solution for $f_x(t)$. Since $\sin u = \cos\left(u - \frac{\pi}{2}\right)$ we can write this solution in the form

$$f_x(t) = \sum_{j=1}^{m} v_j r_j^{t/\theta} \cos\left(\frac{\omega t_j}{\theta} - \varphi_j\right).$$

Here each v_j is a real m-component vector, and φ_j is either 0 or $\pi/2$.

If there are multiple roots among the λ_j, we have only to add a factor t^{k_j} to get the real solution for $f_x(t)$: in this case we have

$$(24) \qquad f_x(t) = \sum_{j=1}^{m} v_j t^{k_j} r_j^{t/\theta} \cos\left(\frac{\omega_j t}{\theta} - \varphi_j\right).$$

Here k_j is zero for every single root λ_j. For an n-fold root $\lambda_{j_1} = \lambda_{j_2} = \ldots = \lambda_{j_n}$ we have to put $k_{j_n} = n-1$. Thus we have the powers

$t^0, t^1, t^2, \ldots, t^{n-1}$ in the corresponding term of $f_z(t)$. Since the formula comprises also the case in which all the roots are single roots (this corresponds to $k_j = 0$ for all j), this formula gives the general real solution of the equation (21).

2. Systematics of the Internal Process. We can now discuss the response of the whole S to a small disturbance of its internal process $\hat{x}(t)$ during an interval $K^{\tau}_{\tau-\theta}$. We know now that this response is determined by the properties of the eigen values of the matrix $(\partial R/\partial \hat{x})_z$. We shall distinguish four mutually exclusive cases related with the information contained in the formula (24) as follows:

(1) all $\lambda_j = 0$:
no information on $f_z(t)$;

(2) all $|\lambda_j| = r_j < 1$, and at least one $\lambda_j \neq 0$:
then $f_z(t) \to 0$ when $t \to \infty$;

(3) only single roots for which $|\lambda_j| = 1$, for others $|\lambda_j| < 1$:
then $f_z(t) \to \sum_j v_j \cos\left(\frac{\omega_j t}{\theta} - \varphi_j\right)$ when $t \to \infty$;

(4) at least one at least double root for which $|\lambda_j| = 1$, or at least one root for which $|\lambda_j| > 1$:
then $f_z(t) \to \infty$ when $t \to \infty$.

Let us study first case (1). In this case we have destroyed our possibility to get information on the trend function f_z by making the substitution $f_z(t) = 0$ from the very beginning. Thus the "result" $f_z(t) = 0$ obtained from the formula (24) in this case does not tell anything but our original substitution.

We must study, in case (1), directly the properties of the matrix

$$\left(\frac{\partial R}{\partial \hat{x}}\right)_z = C \left(\frac{\partial T}{\partial \hat{x}}\right)_z .$$

Remembering the quasi-diagonal form of the matrix T we can represent the matrix $\left(\frac{\partial R}{\partial \hat{x}}\right)_z$ as a composition of the $m_r \times m_s$ matrices

$$\left[\left(\frac{\partial R}{\partial \hat{x}}\right)_z\right]_{rs} = C^{rs} \left(\frac{\partial T_s}{\partial \hat{x}}\right)_z . \tag{25}$$

Here C^{rs} is the $m_r \times n_s$ coupling matrix from the element S_s to the element S_r, and $(\partial T_s/\partial \hat{x})_z$ is the $n_s \times m_s$ matrix composed of the derivatives of T_{sk} with respect to the components $\hat{x}_{sh}(z-\theta)$.

Putting

$$\left[\left(\frac{\partial R}{\partial \hat{x}}\right)_z\right]_{rs} = 0 \text{ for all } r \geq s,$$

the matrix $(\partial R/\partial \hat{x})_z$ will have the following form:

$$\left(\frac{\partial R}{\partial \hat{x}}\right)_z = \begin{pmatrix} 0 & & \overline{R} \\ & 0 & \\ & & \ddots \\ 0 & & 0 \end{pmatrix}.$$

In words: there may be non-vanishing matrix elements only above the main diagonal of the matrix $(\partial R/\partial \hat{x})_z$. It follows that the characteristic matrix has the form

$$\left(\frac{\partial R}{\partial \hat{x}}\right)_z - \lambda \cdot I = \begin{pmatrix} -\lambda & & \overline{R} \\ & -\lambda & \\ & & \ddots \\ 0 & & -\lambda \end{pmatrix}.$$

Hence,

$$\left|\left(\frac{\partial R}{\partial \hat{x}}\right)_z - \lambda \cdot I\right| = (-1)^m \lambda^m.$$

Accordingly, the condition that $\left|\left(\frac{\partial R}{\partial \hat{x}}\right) - \lambda \cdot I\right| = 0$ is in this case equivalent to the condition that $\lambda^m = 0$ or, what is the same, that all the eigen values λ_j are zero.

Now the equations

$$\left[\left(\frac{\partial R}{\partial \hat{x}}\right)_z\right]_{rs} = 0 \text{ for all } r \geq s \tag{26}$$

are valid,

(1.1) if $C^{rs} = 0$ for all $r \geq s$, or

(1.2) if some of these C^{rs} are different from zero there is, however,

$$C^{rs}\left[\frac{\partial T_s}{\partial \hat{x}_s(t-\theta)}\right]_{t=z} = 0 \text{ for all } r \geq s.$$

In case (1.1) there are no circuits of feedback within the whole S. We can see this as follows. If there is a circuit of n elements, say, $S_{r_1} \to S_{r_2} \to \ldots \to S_{r_n} = S_{r_1}$, then all the coupling matrices $C^{r_1 r_2}$, $C^{r_2 r_3}, \ldots, C^{r_{n-1} r_1}$ must be different from zero. The condition for this

is $r_1 < r_2 < r_3 < \ldots < r_n = r_1$ which is a logical contradiction. Accordingly, the condition $C^{rs} = 0$ for all $r \geq s$ is incompatible with the existence of a circuit of feedback within the whole S. On the other hand, if there is no circuit of feedback in S, we can always order the elements $S_1, \ldots, S_N$ so that there will be $C^{rs} = 0$ for all $r \geq s$. Thus case (1.1) is equivalent with the case of no circuits of feedback in the whole S.

In such a case there can be no continued internal process $\hat{x}(t)$ within the whole S. The internal process $\hat{x}(t)$ represents, in this case, a *mechanical reaction* of the system to an external input, and the process stops after the external output, the response, has been given. Since a mechanical reaction is no continued, self-generating process it does not belong to dialectical processes, according to the definition of the latter (see p.146). Neither is there any dialectical contradiction within the whole S, capable only of mechanical reactions.

In case (1.2) some circuit(s) of feedback exist, and thus the whole S is capable of continued, self-generating internal process $\hat{x}(t)$. However, it follows from the condition that the self-generation of the process $\hat{x}(t)$ is at the moment $t = z$ of a passive kind. This is seen as follows. From condition (1.2) we get:

$$(27) \qquad \delta x_{r|s}(z) = C^{rs} \left(\frac{\partial T_s}{\partial \hat{x}_s (t - 0)} \right)_z \delta x_s (z - 0) = 0 \quad \text{for all} \quad r \geq s.$$

Accordingly, the part of the internal process $\hat{x}(t)$, which is due to a feedback does not react in any way at the moment $t = z$ to a small disturbance at $z - 0$. We say that in this case the internal process $\hat{x}(t)$ is *passively self-generating* at the moment z, and that there exists a *latent dialectical contradiction* in the whole S at the moment z.

The existence of case (1.2) shows that from the vanishing of all the eigen values it does not follow that there is no circuit of feedback, and no self-generation, in the system S. On the other hand, if there is no circuit of feedback, then the eigen values are necessarily zero, and we have case (1).

Cases (2), (3), and (4) on p. 156 thus all correspond to cases where at least one circuit of feedback exists in the whole S. In all these cases we call the internal process $\hat{x}(t)$ of the system S *actively self-generating* at the moment $t = z$. We also say that there is, in all these cases, an *active dialectical contradiction* within the whole S at the moment $t = z$.

In case (2) we say that *the goal* or the *direction* $\hat{x}$ is *acceptable* for the whole S at the moment $t = z$. This means the following. If the whole S performs the internal process $\hat{x}(t)$, and if there is a small disturb-

ance of this process during the interval $K^z_{z-\theta}$, the whole S has at the moment $t = z$ the tendency to eliminate the effects of the disturbance, and go back to the process $\hat{x}(t)$.

In case (3) we say that the goal $\hat{x}$ is *indifferent*, or *neutral* for the whole S at the moment $t = z$. This means the following. If S performs the internal process $\hat{x}(t)$, and if there is a small disturbance of this process during the interval $K^z_{z-\theta}$, the system S has at the moment $t = z$ no tendency to eliminate or to increase the effects of the disturbance so influenced on its internal process.

In case (4) we say that *the goal* $\hat{x}$ is *non-acceptable* for the whole S at the moment $t = z$. This means: if S is performing the internal process $\hat{x}(t)$, and if there is a small disturbance of this process during the interval $K^z_{z-\theta}$, the whole S has at the moment $t = z$ the tendency to increase the effects of the disturbance so that its internal process will deviate from the process $\hat{x}(t)$. Accordingly, the internal process, or the action of S is diverging from $\hat{x}$ at the moment $t = z$.

In the latter case it may happen that the whole S will find at some later moment an acceptable goal (either $\hat{x}$ which has became acceptable meanwhile, or a new goal $\hat{\hat{x}} \neq \hat{x}$), On the other hand, it may happen that the goal $\hat{x}$ is forced upon the system S by a larger whole of which S is a part. In such a case the internal process of the whole S cannot diverge from $\hat{x}(t)$ but this goal must be made acceptable by a structural change in S, that is, by a change $C \rightarrow \overline{C}$ of the coupling, so that the new eigen values of the matrix $\partial R/\partial \hat{x} = \overline{C}\partial T/\partial \hat{x}$ will satisfy condition (2) with respect to the process $\hat{x}(t)$. If the goal $\hat{x}$ is not acceptable to the whole S at the moment z, and if no alternative goal exists for S, we can say that there is an *antagonistic contradiction* within the system S at the moment $t = z$. Such a contradiction will in due time necessarily lead to a qualitative change $C \rightarrow \overline{C}$ in the system S.

In other cases the dialectical contradiction is *non-antagonistic*. The internal process of the system S is in case (2) called *self-steering* or *ergodic*, in case (3) *stationary*, and in case (4) *cumulative*, or *anti-ergodic process* at the moment z. Thus we have arrived at the systematics of the internal process shown in Fig. 25.

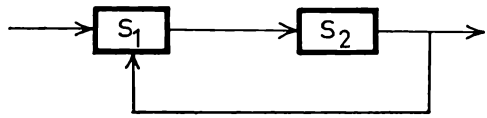


Fig. 24. A two—component feedback system

Oscar Lange[19] called the feedback couplings which make the matrix $(\partial R/\partial \hat{x})_z$ satisfy condition (2) of self-steering the *compensative feedback couplings.*

Exercise. Consider, by the above formalism, the simple system of Fig. 24. Let the coupling coefficients be $c_{12} = c_{21} = 1$. Show that the eigen values of $(\partial R/\partial \hat{x})_z$ are given by

$$\lambda_1 = +\sqrt{\left(\frac{\partial T_1}{\partial \hat{x}_1}\right)_z \left(\frac{\partial T_2}{\partial \hat{x}_2}\right)_z}, \quad \lambda_2 = -\sqrt{\left(\frac{\partial T_1}{\partial \hat{x}_1}\right)_z \left(\frac{\partial T_2}{\partial x_2}\right)_z}.$$

These are real, if the derivatives have the same sign. In such a case the feedback is called *positive.* The eigen values are imaginary, if the derivatives have the opposite signs. The feedback is then called *negative.* The condition of self-steering at the moment z is

$$\left|\left(\frac{\partial T_1}{\partial \hat{x}_1}\right)_z\right| \cdot \left|\left(\frac{\partial T_2}{\partial \hat{x}_2}\right)\right| < 1.$$

Show that the condition of self-steering can in this case also be expressed as the condition of the negative-definiteness of the matriix

$$\left(\frac{\partial R}{\partial \hat{x}}\right)_z^2 - I.$$

19. Oskar Lange, *Wholes and Parts*, London 1965. It may be appropriate here to make a note of the differences the reader will find between the exposition given here in § 1 and § 2, on the one hand, and the exposition of these matters given by Lange. The innovations of my exposition, compared with Lange's can be listed as follows: 1) the distinction between the trend function (f_z or F_z) and the corresponding difference process Δx; this is important, because it indicates that 2) the properties of the matrix $(\partial R/\partial \hat{x})_z$ determine only the local properties of the process $\hat{x}(t)$ in a future neighbourhood $K_z^{z+\varepsilon}$ of the moment z; this in turn 3) leads to a systematics and definitions of self-generating processes (see Fig. 25) differing from those given by Lange; it also indicates that 4) the properties of the derivatives $(\partial^n R/\partial \hat{x}^n)_z$, $n = 1,2,\ldots$, taken at the point z, determine only the local properties of a 'large' difference process $\Delta x(t)$ (see § 2.2.), i.e. the properties in a future neighbourhood $K_z^{z+\varepsilon}$ of z. One can also mention 5) the replacemant of Lange's formalism of quasi-diagonal matrices X and Y for the representation of internal processes, by a formalism using solely the vectors x and y; this allows 6) a separation of the roles of the matrices C and $\partial T/\partial x$ as determinants of the local properties of the internal process; this in turn makes possible 7) the distinction between the non-existence of feedback couplings (case 1.1) and the non-existence of active self-generation or, in other words, the distinction between the mechanical reactions and the passively self-generating processes. Finally, 8) the coupling coefficients c may have here any real values, while in Lange's exposition their values were restricted to 0 and 1.

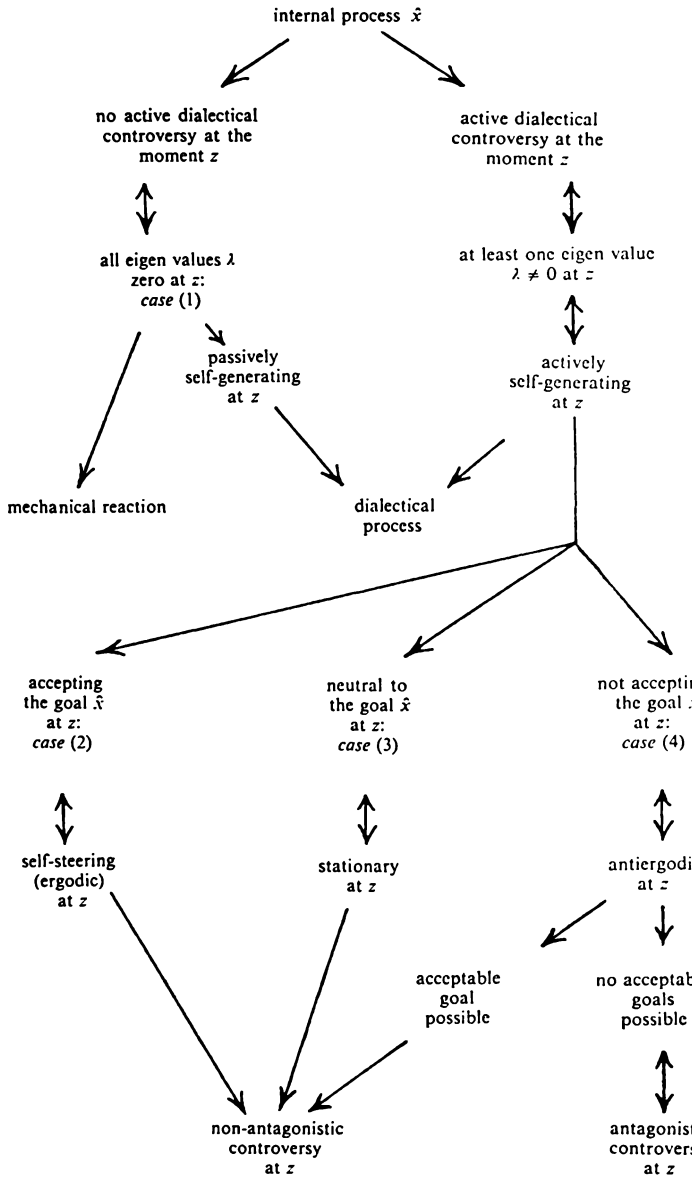


Fig. 25. The systematics of the internal process

3. Solution for Gradual Effect. The results do not change when we go to the cases of gradual effect. We shall here point only to some differences in comparison with the case of sudden effect.

For a gradual effect we have a matrix $\partial R/\partial \hat{x}$ for each value of the magnitude τ:

$$\left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z = \left[\frac{\partial R(\tau)}{\partial \hat{x}(t-\tau)}\right]_{t=z}.$$

The defining equations of the trend function f_z then become:

$$(28)\quad \begin{cases} f_z(t) = \sum_{\tau_1}^{0} \left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z f_z(t-\tau) & \text{(step-wise effect),} \\ f_z(t) = \int_{\delta}^{0} \left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z f_z(t-\tau)\,\mathrm{d}\tau & \text{(continuous effect).} \end{cases}$$

The substitution $f_z(t) = w\lambda^{t/0}$ now leads to the characteristic equations

$$(29)\quad \begin{cases} \left| \sum_{\tau_1}^{0} \left(\frac{\partial R(\tau)}{\partial \hat{x}}\right) \lambda^{-\tau/0} - I \right| = 0 & \text{(step-wise effect),} \\ \left| \int_{\delta}^{0} \left(\frac{\partial R(\tau)}{\partial \hat{x}}\right) \lambda^{-\tau/0}\,\mathrm{d}\tau - I \right| = 0 & \text{(continuous effect).} \end{cases}$$

2 / The Study of Internal Contradictions 'In the Large': Purposiveness and Ergodicity

Let us now study a difference process $\Delta x(t) = x(t) - \hat{x}(t)$ "in the large". In other words, we shall again assume that $x(t)$ and $\hat{x}(t)$ are two solutions of the Inner Law of Motion, (II) on p. 148, but now the initial segment $(x-\hat{x})\big|K_{z-0}^{z}$ is not composed of small quantities of the first degree.

Assuming the necessary conditions of analyticity and convergence are fulfilled, we can again use the Taylor series (12) for the difference process. However, we must now take into consideration the higher terms too. Accordingly, to express the course of the difference process in some future neighbourhood $K_z^{z+\epsilon}$ of the time point $t = z$, we must now replace the equations (18) on p. 153 by the equations

$$(30)\begin{cases}\Delta x(t) = \left(\frac{\partial R}{\partial \hat{x}}\right)_z \Delta x(t-0) + \frac{1}{2!}\Delta x(t-0)' \left(\frac{\partial^2 R}{\partial \hat{x}^2}\right)_z \Delta x(t-0) + \\ + \ldots \text{ for } t \in K_z^{z+\varepsilon} \qquad \text{(sudden effect)},\\ \Delta x(t) = \sum_{\tau_1}^{0}\left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z \Delta x(t-\tau) + \frac{1}{2!}\sum_{\tau_1}^{0}\Delta x(t-\tau)'\left(\frac{\partial^2 R(\tau)}{\partial \hat{x}^2}\right)_z \\ \Delta x(t-\tau) + \ldots \text{ for } t \in K_z^{z+\varepsilon} \qquad \text{(step-wise gradual effect)},\\ \Delta x(t) = \int_{\delta}^{0}\left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z \Delta x(t-\tau)\mathrm{d}\tau + \frac{1}{2!}\int_{0}^{0}\Delta x(t-\tau)' \left(\frac{\partial^2 R(\tau)}{\partial \hat{x}^2}\right)_z \\ \Delta x(t-\tau)\,\mathrm{d}\tau + \ldots \text{ for } t \in K_z^{z+\varepsilon} \qquad \text{(continuous gradual effect)}.\end{cases}$$

We can now introduce the *trend function "in the large"* defined by

$$(31)\begin{cases}F_z(t) = \left(\frac{\partial R}{\partial x}\right)_z F_z(t-0) + \frac{1}{2!}F_z'(t-0)\left(\frac{\partial^2 R}{\partial \hat{x}^2}\right)_z F_z(t-0) + \\ + \ldots \textit{ for all } t > z \qquad \text{(sudden effect)},\\ F_z(t) = \sum_{\tau_1}^{0}\left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z F_z(t-\tau) + \frac{1}{2!}\sum_{\tau_1}^{0}F_z'(t-\tau)\left(\frac{\partial^2 R(\tau)}{\partial \hat{x}^2}\right)_z \\ F_z(t-\tau) + \ldots \textit{ for all } t > z \qquad \text{(step-wise gradual effect)},\\ F_z(t) = \int_{\delta}^{0}\left(\frac{\partial R(\tau)}{\partial \hat{x}}\right)_z F_z(t-\tau)\,\mathrm{d}\tau + \frac{1}{2!}\int_{\delta}^{0}F_z'(t-\tau)\left(\frac{\partial^2 R(\tau)}{\partial \hat{x}^2}\right)_z \\ F_z(t-\tau)\,\mathrm{d}\tau + \ldots \textit{ for all } t > z \qquad \text{(continuous gradual effect)},\end{cases}$$

and by the initial condition that

$$(32) \qquad F_z\big|K_{z-\theta}^z = \Delta x\big|K_{z-0}^z.$$

Due to the equations (30) and (31) and to the initial condition (32), the functions F_z and Δx coincide in the nearest future after the moment z, viz. for $t \in K_z^{z+\varepsilon}$:

$$F_z(t) = \Delta x(t) \text{ for } t \in K_z^{z+\varepsilon}.$$

However, due to the differences of the derivatives in the equations 12), p. 150, on the one hand, and the equations (31), on the other, viz.

$$\left(\frac{\partial R}{\partial \hat{x}}\right)_t - \left(\frac{\partial R}{\partial \hat{x}}\right)_z \neq 0, \left(\frac{\partial^2 R}{\partial \hat{x}^2}\right)_t - \left(\frac{\partial^2 R}{\partial \hat{x}^2}\right)_z \neq 0 \text{ etc.},$$

the functions F_z and Δx may differ from one another for the values $t > z + \epsilon$.

Since we have assumed the convergence of the series in (31), the function F_z can be approximated in successive approximations by the functions $F_z^{(1)}, F_z^{(2)}, F_z^{(3)}$ etc., the function $F_z^{(n)}$ being defined by the polynomial of the degree n including only the first n derivatives of R in the equations (31), and satisfying the respective initial condition.

If all the values of the initial segment $(x-\hat{x})\big|K_{z-0}^z$ are small quantities of the first order, the function F_z is reduced to the function f_z:

$$F_z(t) \longrightarrow f_z(t) \quad \text{when} \quad \Delta x\big|K_{z-0}^z \longrightarrow 0. \tag{33}$$

Thus the trend function F_z can be considered as an extension of the trend function f_z for non-infinitesimal, finite values of $\Delta x\big|K_{z-0}^z$.

We can use the function F_z to indicate the trend of the internal process $\hat{x}(t)$ in all cases in which this process is self-generating, whether actively or passively. Obviously, the behaviour of the trend function F_z depends

1° on all the derivatives $(\partial^n R/\partial \hat{x}^n)_z$ (sudden effect), or $(\partial^n R(\tau)/\partial \hat{x}^n)_z$ (gradual effect), for $n = 1,2,3, \ldots$, and

2° on the initial segment $\Delta x\big|K_{z-0}^z$, representing the disturbance of the internal process.

We have the following three possible cases as to the asymptotic behaviour of the function F_z.

First, it may happen that $F_z(t)$ tends to zero when t goes to infinity. We say then that the initial segment $\Delta x\big|K_{z-0}^z$ belongs to the *domain of ergodicity* of the process $\hat{x}(t)$ at the moment z. The whole S has then, at the moment $t = z$, a tendency to eliminate the effects of the disturbance $\Delta x\big|K_{z-0}^z$. Obviously, it follows from the respective definitions that if the process $\hat{x}(t)$ is self-steering at the moment z, then it has a certain domain of ergodicity at this moment.

Secondly, it may happen that $F_z(t)$ approaches a constant, or an oscillation function of t, when t goes to infinity. We say then that the initial segment $\Delta x\big|K_{z-0}^z$ belongs to the *domain of stationarity* of the process $\hat{x}(t)$ at the moment z. In this case the whole S has, at the moment z, no tendency to eliminate or to increase the effects of the disturbance $\Delta x\big|K_{z-0}^z$. Obviously, if the process $\hat{x}(t)$ is stationary at the moment z, it has a domain of stationarity at this moment, such that the zero disturbance $\Delta x\big|K_{z-0}^z = 0$ belongs to it. If, on the other hand, the process $\hat{x}(t)$ is self-steering at the moment z, it may have

a domain of stationarity at the boundary of the domain of ergodicity. In the latter case the domain of stationarity does not include the zero disturbance which belongs to the domain of ergodicity in this case.

Thirdly, $F_z(t)$ may go to infinity when t increases sufficiently. Then we say that the initial segment $\Delta x \big| K^z_{z-\theta}$ belongs to the *domain of anti-ergodicity* of the process $\hat{x}(t)$ at the moment z. The whole S then has a tendency to increase the effects of the disturbance $\Delta x \big| K^z_{z-\theta}$. If the process $\hat{x}(t)$ is cumulative at the moment z, it has at this moment a domain of anti-ergodicity which includes the zero disturbance. If the process $\hat{x}(t)$ is self-steering or stationary at the moment z, it may still have a domain of anti-ergodicity composed of large enough disturbances $\Delta x \big| K^z_{z-\theta}$.

Obviously any disturbance $\Delta x \big| K^z_{z-\theta}$ of a self-generating process $\hat{x}(t)$ belongs either to the domain of ergodicity, or to the domain of stationarity, or to the domain of anti-ergodicity of the process $x(t)$ at every moment z of the calendar K of the system S. We are now interested only in the domains of ergodicity of a self-steering process, and shall have a glance at them in the following.

The domain of ergodicity is obviously a characteristic of a self-generating process $\hat{x}(t)$ occurring in a whole S, and it is associated with a certain moment z of time in the calendar K of the system S. For different wholes S, or for different self-generating internal processes $\hat{x}(t)$ of the same whole S, or for different moments of time during the course of the same process $\hat{x}(t)$, there are different domains of ergodicity. Thus we can denote a definite domain of ergodicity by the symbol $D_{\hat{x},S}(z)$.

It follows from the definition of ergodicity that the domain $D_{\hat{x},S}(z)$ is a subset of the vector function space $F(K^z_{z-\theta}, R^m)$ composed of all the possible disturbances $\Delta x \big| K^z_{z-\theta}$:

$$D_{\hat{x},S}(z) \subset F(K^z_{z-\theta}, R^m) = \{\Delta x \big| K^z_{z-\theta}\}. \tag{34}$$

Here R^m is the m-fold cartesian product of the set R of the real numbers with itself, and R^m represents the range of values of the real, m-component vectors $\Delta x(t)$. The domain $D_{\hat{x},S}(z)$ of a self-steering process $\hat{x}$ contains the zero disturbance.

We need a measure of the largeness of the domain of ergodicity. The construction of such a measure is a simple thing, if S is a digital system. Then the time interval $K^z_{z-\theta}$ is a sequence of a finite number

of moments of times represented by successive integers, θ being an integer too:

$$K^z_{z-0} = \{z-\nu\,;\ \nu = 1,2,\ldots,\theta\} \quad \text{(digital system)}.$$

An arbitrary disturbance $\Delta x\big|K^z_{z-0}$ is then represented by an $m \times \theta$ matrix Δ:

$$\Delta x\big|K^z_{z-0} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \ldots & \Delta_{1\theta} \\ \Delta_{21} & \Delta_{22} & \ldots & \Delta_{2\theta} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \Delta_{m1} & \Delta_{m2} & \ldots & \Delta_{m\theta} \end{bmatrix} = \Delta.$$

Here $\Delta_{i\nu} = \Delta x_i(z-\nu)$.

The matrices Δ span an $(m\theta)$-dimensional real vector space, $V_{m\theta}$, where the norm square of each vector Δ is given by

$$|\Delta|^2 = \mathrm{tr}\Delta'\Delta = \mathrm{tr}\Delta\Delta' = \sum_{i,\nu} \Delta^2_{i\nu}.$$

We call $|\Delta|$ the norm of the disturbance Δ. Each domain $D_{\hat{x},S}(z)$ of ergodicity of a digital system S is a finite domain in the vector space $V_{m\theta}$, containing the origin (see Fig. 26). A measure of the 'largeness' of $D_{\hat{x},S}(z)$ would literally be the volume V_D of its domain in $V_{m\theta}$. However, we are more interested in the average magnitude of the disturbances belonging to $D_{\hat{x},S}(z)$. Accordingly, we define the largeness, or the *width of the domain of ergodicity* as the average value of the norm square $|\Delta|^2$:

$$w(D_{\hat{x},S}(z)) = \int_{V_D} |\Delta|^2 \,\mathrm{d}\Delta_{11}\ldots\mathrm{d}\Delta_{m\theta} \Big/ \int_{V_D} d\Delta_{11}\ldots\mathrm{d}\Delta_{m\theta}. \tag{35}$$

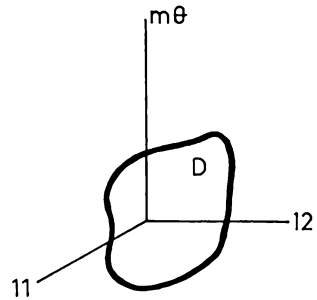


Fig. 26. The domain D of ergodicity for a digital system.

For a general system S each component Δx_i of each disturbance $\Delta = \Delta x \big| K^z_{z-\theta}$ is a function of time defined in the interval $(z-\theta, \theta)$ of the real axis representing the time. The norm square of a disturbance then becomes

$$\left|\Delta\right|^2 = \sum_{i=1}^{m} \int_0^{\theta} \Delta x_i^2 (z-\nu)\, d\nu .$$

If $\Sigma = \{\sigma\}$ is the index set of the set $\{\Delta x \big| K^z_{z-\theta}\}$ of all the m-component vector functions from $K^z_{z-\theta}$ to R^m, we have to define a measure μ in Σ, that is: a function μ from $F(\Sigma)$ to the set of non-negative real numbers, so that $\mu\left(\bigcup_{k=1}^{\infty} \Sigma_k\right) = \sum_{k=1}^{\infty} \mu(\Sigma_k)$ for any sequence of disjoint sets $\Sigma_1, \Sigma_2, \ldots \varepsilon F(\Sigma)$, and $\mu(\Phi) = 0$ for the empty set Φ. Then we can define the width of the domain of ergodicity by

$$w(D_{\hat{x},S}(z)) = \int_{D_\Sigma} \left|\Delta_\sigma\right|^2 d\mu(\sigma) \Big/ \int_{D_\Sigma} d\mu(\sigma). \tag{36}$$

Here D_Σ is the subset of Σ composed of the indeces of the disturbances belonging to $D_{\hat{x},S}(z)$:

$$D_\Sigma = \{\sigma;\ \Delta_\sigma \varepsilon D_{\hat{x},S}(z)\} \subset \Sigma .$$

Another characteristic which is interesting in connection with the ergodicity of a self-steering process is the strength of ergodicity. We define the *strength of ergodicity* of a self-steering process $\hat{x}(t)$ in a system S at a moment z by

$$E_{\hat{x},S}(z) = \left|\frac{df_z(t)}{dt}\right|_{t=z} . \tag{37}$$

Thus the strength of ergodicity indicates the speed at which the whole S begins the elimination of the effects of a small disturbance $\Delta x \big| K^z_{z-\theta}$ at the moment $t = z$.

Substituting into the formula of $E_{\hat{x},S}(z)$ the general solution (24) for the function $f_z(t)$ in the case of sudden effect, we get:

$$E_{\hat{x},S}(z) = \left| \sum_j \nu_j k_j z^{k_j - 1} (1 - \delta_{0k_j})\, r_j^{z/\theta} \cos\left(\frac{\omega_j z}{\theta} - \varphi_j\right) + \right.$$

$$\left. + \frac{1}{\theta} \sum_j \nu_j z^{k_j} r_j^{z/\theta} \log r_j \cos\left(\frac{\omega_j z}{\theta} - \varphi_j\right) - \right.$$

$$-\frac{1}{\theta}\sum_j v_j z^{k_j} r_j^{z/\theta}\,\omega_j \sin\left(\frac{\omega_j z}{\theta}-\varphi_j\right)\Bigg| =$$

$$=\left|\sum_j v_j z^{k_j-1} r_j^{z/\theta}\left\{k_j(1-\delta_{0k_j})\cos\left(\frac{\omega_j z}{\theta}-\varphi_j\right)+\right.\right.$$

$$\left.\left.+\frac{1}{\theta}\,z\log r_j\cos\left(\frac{\omega_j z}{\theta}-\varphi_j\right)-\frac{1}{\theta}\,z\omega_j\sin\left(\frac{\omega_j z}{\theta}-\varphi_j\right)\right\}\right|$$

$\longrightarrow 0$ when all $r \longrightarrow 0$.

Here the 'Kronecker's delta symbol' δ_{0k_j} is defined as having the value 1, if $k_j = 0$, and being otherwise zero. Thus the first term in the parentheses vanishes if all the roots λ_i are single roots. In such a case we get the further result that

$$E_{\hat{x},S}(z) \longrightarrow 0 \quad \text{when all} \quad \lambda_j \longrightarrow +1 \text{ (for non-multiple roots).}$$

Indeed, if all $\lambda_j \longrightarrow +1$, then all $\log r_j$ and all ω_j approach zero, and so approach the second and the third term in the parentheses. (The limiting case that all λ_j are $+1$, however, represents itself as an m-fold multiple root, and must thus be excluded, since the first term does not then vanish.)

The results related with ergodicity, similar to those above, were given the following interpretation by Oskar Lange[20]. The cybernetic notion of ergodicity gives a possible causal explanation for the *purposive* behaviour. Indeed, it is an essential characteristic of purposive behaviour that the purposive being, when disturbed, returns back to follow his goal, as if directed by an inner force called his 'own will'. In reality this direction is determined by the internal structure of the system in question, i.e. by the coupling matrix C and by the matrix $\partial T/\partial x$. When the eigen values of $(\partial R/\partial x)_{\hat{x}} = C(\partial T/\partial x)_{\hat{x}}$ all approach zero, the elimination of disturbance is slow, as is shown by the above formula for $E_{\hat{x},S}(z)$. It might happen that even the width of the domain of ergodicity is small then, so that small disturbances already would suffice to direct the system's behaviour from its earlier goal $\hat{x}$. Both of these situations

20. Oskar Lange, *Wholes and Parts*, London 1965. Lange gave no definition for the width or the largeness of the domain of ergodicity, and he did not calculate a correct value for strength or speed $E_{\hat{x},S}(z)$ (cf. ibid., p. 66), thus concluding erroneously that $E \to 0$ when all $|\lambda_j| \to 1$. However, these minor inexactitudes do not influence the main results of Lange related with ergodicity. For other differences between Lange's work and the present theory see the footnote on p. 159.

are characteristic of a young living being, whose purposive bahaviour is still uncertain. On the other hand, the 'death' of the system means its permanent transformation to a decaying, anti-ergodic state for which $|\lambda| > 1$. Obviously, there is a transitory period characterized by $|\lambda| = 1$. For an 'ageing' system we could expect the domain of ergodicity to be small again: the strength of the purposive behaviour of an 'ageing' system is decreased. On natural selection in the phylogenetic development Lange says: "only those ergodic processes of development (and the corresponding systems) remain which are resistant to a high degree against disturbances, i.e. processes with a large domain of ergodicity and speedy disappearance of disturbances." (Lange, ibid., p. 68).

3 § On Future Development and Open Problems of Cybernetic Theory

1 / The Need for a Theory Extending Over the Successive Phases of Self-Generating Dialectical Process: The Problem of Complication

A limitation of the present cybernetic theory of self-generation is striking: this theory, presented above in §1 and §2, is restricted to a momentary analysis. The present theory tells only in which phase the self-generating process is at a certain moment of time, and within a certain interval of time around a given point of time.

Accordingly, the existing theory tells whether the internal process of a given system is, at a certain moment, ergodic, or anti-ergodic, or stationary, or mechanical. If the internal process is in an ergodic phase this means that the system in question is in a process of purposive, gradual development toward a certain goal. If the process is in an anti-ergodic phase this means that the system is either changing its goal or, if no transition to an ergodic process is possible, developing toward an open antagonistic contradiction as a consequence of which the structure (and the process) of the system will undergo a qualitative change in the near future.

Thus we can only say, on the basis of the present theory and if a complete information is available, whether a cybernetic whole whose structure is known is undergoing an evolutionary or a revolutionary phase of its internal, dialectical process at a given moment. The existing theory

does not tell how the evolutionary and revolutionary, gradual and qualitative changes will follow each other in the course of the dialectical process. We lack a cybernetic theory extending over the successive phases of the self-generating dialectical process. Or, to use the classical terminology of dialectical materialism, we still lack a mathematical theory which could tell how the accumulation of certain 'quantities' will produce new qualities ('*transformation of quantities into qualities*').

The problem mentioned seems closely connected with the problem of complication. With increasing structural and functional complexity cybernetic systems seem to acquire qualitatively new properties, which the simpler systems do not possess. For instance, below a certain level of complication, as is shown in the theory of self-reproduction (See Appendix of Volume II), material systems are able to produce only such other systems which are less complicated than they themselves are (degenerative evolution). However, beginning with a certain level of complication, self-generation, and even an evolution creating completely new qualities, becomes possible. With such phenomena in mind John von Neumann was inclined to connect the problem of complication with the problem of the creation of new qualities:

"The discussions so far have shown that high complexity plays an important role in any theoretical effort relating to automata, and that this concept, in spite of its prima facie quantitative character, may in fact stand for something qualitative — for a matter of principle." (J.v. Neumann, The general and logical theory of automata, p. 25, in L.A. Jeffress (Ed.), *Cerebral Mechanisms in Behaviour*, Hafner Co 1967).

It is obvious that the theory of self-generation must proceed to a theory in which the self-generated increased complexity of the developing systems produces new qualities in the course of increasing complication. First then we can consider the succession of gradual and qualitative changes of the dialectical process by means of cybernetic theory.

2 / The Need For Spatial Localization of Cybernetic Systems: Cellular or Tessellation Models

In the above, in § 2.5, we have explicitly introduced time to the description of cybernetic systems. Of course not only the time coordinate but also the spatial coordinates, indicating the position of the system and of its parts in space, should be explicitly introduced in order to describe completely the characteristics of cybernetic systems as material

beings existing in space and time. This will certainly be a line of development in future cybernetic theory.

The necessity of an explicit spatial description of cybernetic systems has been indicated by some erroneous conclusions based on a system's description where the explicit spatial description is missing. As emphasized by the criticism of the so called Rosen paradox, for instance, the neglect of the spatial properties of cybernetic systems may lead to an apparent paradox in the notion of self-reproducing systems[21].

The examples now existing of the explicit introduction of space coordinates in cybernetic theory are called cellular or tessellation models. We shall meet an example of these models in Appendix of Volume II. In these models physical space is divided into cells, each of them having a certain number of possible states (in a model of homogeneous space each cell has, of course, the same set of possible states). A cybernetic system — for instance, an automaton — can now be described as a spatial distribution of certain kinds of states over the cells.

It is possible that space coordinates must be introduced in future cybernetic theory not only in the context of certain particular type of problems, like self-reproduction, but in general. This would mean a mathematical metamorphosis of cybernetic theory, starting, for instance, with the introduction of a cellular space as a first approximation to a spatial localization of cybernetic systems. The von Neumann construction of self-reproducing automata in fact suggests that the problem of spatial localization and the problem of complexity are two aspects of one and the same promlem, to be solved in future *topological cybernetics.*

3 / The Need for Realistic Probabilism: Thermodynamic models and Error Theory

The views of the philosophical role of probability in science differ widely depending on whether you consider it from the point of view of idealism or from the point of view of materialism. Therefore, we must carefully distinguish between two philosophies of probability in science, the first of which is incompatible with dialectical materialism, and with any sound scientific theory, while the second is what is actually likely to be realized in cybernetics as well as in many other branches

21. See, e.g., L. Löfgren, Kinematic and tessellation models of self-repair, *Biological Prototypes and Synthetic Systems*, Vol. 1, New York 1962, p. 362.

of science. These two philosophies of probabilism in science could be called idealistic probabilism and realistic probabilism, respectively.

Idealistic probabilism suggests the replacement of the deterministic chains of cause and effect by probabilistic notions everywhere in scientific theory, as a matter of principle. Idealistic probabilism thus denies the existence of causal determinism.

Realistic probabilism holds to causal determinism as a fundamental principle of science. It considers the introduction of probabilities just as a realistic device, to take into account our actual ignorance, our inadequate knowledge of some factors influencing the phenomena we are studying.

Idealistic probabilism is one of the many forms that the positivistic pursuit has taken in the bourgeois philosophy of science in our day. Encouraged by the introduction of probabilities into quantum theory it is often stated that causal determinism, such as is involved in scientific materialism, is 'outdated'. Quantum theory has proved, one says, that science cannot tell the ultimate truth about the existing world, that the chains of cause and effect cannot explain the phenomena of our world. So religion is necessary to fill the gap, after all.

This is a wishful interpretation of quantum theory. In reality there is nothing in quantum theory which would deny causal determinism as a fundamental principle of science. Probabilism in quantum theory is due to certain restrictions we meet when performing measurements in the very small dimensions of the atomic world (expressed in the so-called relation of uncertainty). These restrictions are taken into account in a realistic physical theory by introducing probabilities. Thus it is realistic probabilism, and not idealistic probabilism, which is inherent in modern quantum theory.

The situation is the same in cybernetics. We can never accept idealistic probabilism in the context of cybernetic theory. Thus the basic chains of cause and effect, which form the foundations of cybernetic theory in the form of input-output relations, will never be 'outdated' as the essential fundament of cybernetic theory. On the other hand, realistic probabilism is very likely to appear in future cybernetic theory too.

Von Neumann's ideas on future cybernetic theory here deserve particular attention. Von Neumann remarked that in many realistic situations where we have to apply cybernetic theory we do not know exactly which input comes to the system at which time. Our ignorance is best expressed by saying that we can only tell something of the probability $p(x)$ of a certain input x to appear (at a certain time t, or on

average over an interval of time, for instance). In fact we have followed such a theoretical scheme all the time when discussing the input organization (and the output organization, for that matter) on pp. 125—132 above. This approach can be developed by developing a systematical probabilistic description of the *environment* of the system in the form of a statistical theory. So we come to a statistical theory of information which, due to its obvious connections with entropy and other thermodynamic concepts, is likely to be *a theory of the thermodynamical type* (cf von Neumann, *Theory of Self-Reproducing Automata*, Illinois and London 1966 p. 62—63). von Neumann says in his prophecy: "The statistical variables of the automaton's mileu will, of course, be somewhat more involved than the standard thermodynamical variable of temperature, but they will probably be similar in character. — — I will not go into the details of this, but I would like to emphasize that this thermodynamical link is probably quite a close one."

The description of the *environment* of the system being 'probabilized' by means of a systematical, thermodynamical type of theory, the probabilities $p(x)$ of the inputs x so defined will of course creep into the description of the system too. If the inputs are determined up to a certain probability, so are the outputs and the inner states of the system. However, all the other probabilities in such a theory are strict consequences of the input probabilities $p(x)$, obtained when the latter are introduced to the strictly deterministic input-output and state transition formulae. Thus the 'thermodynamization' of the description of the environment does not influence the strictly deterministic discussion of the functioning of the system in any way. We only have a strictly deterministic system located in a probabilistic environment, as a consequence of which the output of the system also appears to be statistically distributed.

One further step toward a realistic probabilism in cybernetic theory would be the introduction of *cybernetic error theory* into the description of the actual functioning of the components. Such a step could be motivated as follows. Usually the components we consider in a cybernetic system are fairly large material objects. The correct functioning of such a component, described by the action operator T of this component, is a consequence of certain cause-and-effect links in the inner structure of the component which we do not explicitly consider in the theory. Thus many factors, like common wear and tear, which are not explicitly taken into account in the theory, can still effect the operator T. In other words, there is the possibility that the correct functioning of

the component ceases, and the operator T suddenly is no more what it used to be. Such a possibility can be indicated by introducing a certain life time for each component or, what amounts to the same, by introducing certain probabilities by which the action operator T, when applied to a fixed input x, will give different values of y. So, for instance, if $T(x) = y_0$ indicates the correct functioning of the component in question, we could postulate that $T(x)$ may have the value y_0 by a certain probability, say 0.95. The remaining part 0.05 of the probability mass could be distributed over possible values $y \neq y_0$, each of which indicates a certain type of *malfunction* of the component.

One must again emphasize that neither the suggested first, thermodynamic step nor the second, error-theoretical step of probabilism will in any way change the foundation of cybernetic theory on strictly causal determinism. There will for ever be the *strictly causal recursion* of the output of the system to the inner structure and the input in cybernetic theory. The probabilities we introduce into the theory will have no significance in principle: they only express our actual ignorance of this or that part of the causal process we are studying. Thus both suggested steps belong to the realm of realistic, not idealistic, probabilism.

4 / One Further Need in Future Cybernetic Theory: A Theory of Sensitive Systems

One important further need in future cybernetic theory has been in the existing theory so little appreciated that we must restrict ourselves just to mention it here. It is the need for a theory of sensitive systems. By a sensitive system we mean a cybernetic system (or a whole), where the coupling parameters c as well as the action operators T may change in the course of the internal process of the system. The cortex of human brain obviously is an example of such a sensitive system, and a cybernetic theory of such systems would have much application in social science too. But the theory of sensitive system may be closely connected with the theory of qualitative change, and thus with the cybernetic theory of complication (see above, p. 169).

5 / What Will Be Preserved of Present Cybernetic Theory?

Whatever innovations there will appear in future cybernetic theory the solid materialistic foundation of cybernetics is there, for sure, to stay. What does this imply?

It implies

(1) preservation of the nature of cybernetic systems as *material beings localized in space and time:* this is expressed by the term M_b in the definition of cybernetic system,

(2) presevation of *the fundamental causal recursion* in cybernetic systems: this is expressed by the causal relation R_c expressing the recursion of the output of the system to its input and inner structure.

Thus the materialistic foundation of cybernetic theory can be expressed by the representation of cybernetic system as a combination $S = (M_b, R_c)$ where $R_c = \mathrm{Ch}\, M_b$. The essentials of this foundation, which were studied in more detail in Chapter II above, will hardly be subjected to any principal alterations in any future theory though an axiomatization may be developed.

If the positivistic philosophy of science is still able to hold its position in the Western world for some time, there will of course be idealistic interpretations of cybernetics too, just as there are at the present. The idealistic interpretations of cybernetics will emphasize, just as they emphasize at present, the purely formal nature of the objects of cybernetics. Idealistic interpretations will also seek after possibilities, as they are seeking at the present, to deny the fundamental *causal recursion* dominating cybernetic theory. However, these linguistic-idealistic interpretations of cybernetic theory will hardly have influence on the actual progress of cybernetics.

The present cybernetic theory is not only any materialistic but is opposed to mechanistic materialism. This fundamental characteristic of cybernetic theory can be expected to be preserved in future theory as well. This means

(3) preservation and further development of the nature of fundamental cybernetic theory as a *theory of self-generation* of material beings actively changing themselves and their environment in a self-generating dialectical process, and

(4) preservation of the typical methodological approach of cybernetics, viz. *proceeding from wholes to parts.*

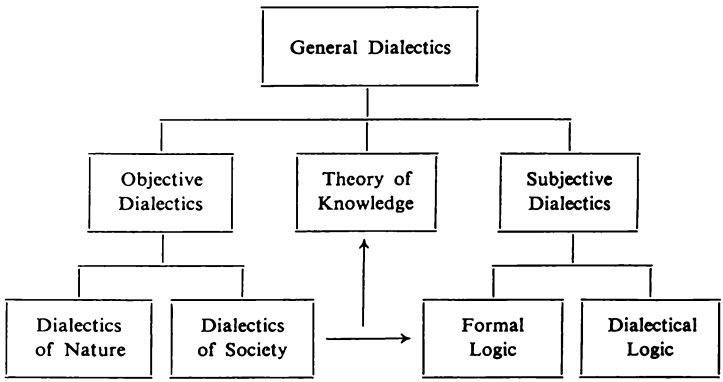
One could say that both of these characteristics will not only be preserved but become more pronounced in future cybernetic theory. In fact we have seen that there are considerable defects in the present theory of self-generation: at present, we cannot yet follow the develop-

ment of self-generating dialectical processes over successive phases in mathematical theory, but are so far confined to a momentary phase of development. In future theory this restriction should be overthrown. One should eventually develop a cybernetic theory where the very fundamental laws of the self-generating systems are self-generating themselves: such a progress from self-generating systems to self-generating fundamental laws is possibly connected with the problem of complication in cybernetic theory.

CHAPTER IV:

The Cybernetic Model of Rational Actor

Georg Klaus, in his book on modern logic[22], gives the following table to indicate the systematics of general dialectics:



The subject of the present book in the preceding has been *objective dialectics*. However, we shall now make an excursion which extends also to the department of subjective dialectics entitled 'Formal Logic'.

Turing constructed a model of a 'rational actor' capable of performing the logical steps required in correct computation. As 'computation', as characterized by Turing, is the prototype of all kinds of logical deduction, his model in fact is a model of beings capable of logical action.

22. Georg Klaus, *Moderne Logik*, Berlin 1972.

1 § The Turing Machine as a Cybernetic System

1 / The Human Cognitive System

We can approach the idea of the Turing machine best if we start with the feedback system of the human cognitive system. By 'cognitive system' we mean the total system composed of the central nervous system, the receptors, and the effectors. The main channels of coupling in this system are indicated in Fig. 27.

The main channels of coupling are indicated by heavy lines. These lines show the main course of the input-output processes in the system. The receptors first receive inputs from the environment of the system, then send outputs to the central nervous system, which sends outputs to the effectors. The effectors finally influence the environment by sending outputs. This line of channels forms the *simple stimulus-response channel* of impulses. Let us remark that the inputs come to this channel both from the outer environment of the organism and from the environment of the cognitive system within the organism: the receptors receive inputs from both environments, the external and the internal. Likewise the effectors influence by their outputs both the external and the internal environment of the body.

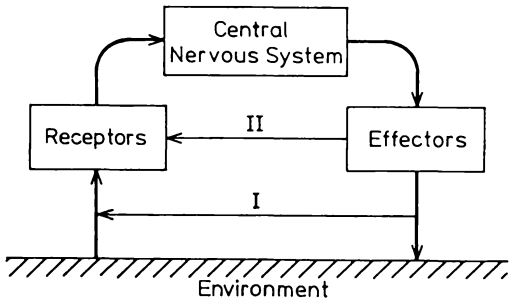


Fig. 27.

In addition to the simple stimulus-response channel the cognitive system contains important feedbacks. One of the feedback channels, marked by Roman numeral I in Fig. 27, may transmit to the receptors messages on all the actions of the effectors. The receptors obtain

these messages by inspecting the influences of the effectors on the environment of the cognitive system (both external nad internal). Therefore feedback channel I is drawn to begin from the output channel of the effectors and to enter the environmental input channel of the receptors. The other feedback channel (II) comes to the receptors directly from the effectors. Through it comes the direct mechanical or chemical influence of the muscles and the glands on the position and on the physico-chemical state of the receptors (for instance, moving of the eyes by eye muscles or by turning the head).

2 / Description of the Turing Machine

Since the Turing machine[23] can be understood as a simple, idealized model of the human cognitive system, it involves parts which correspond to the three main parts of the cognitive system. In the Turing machine a finite automaton *A* corresponds to the central nervous system. Device *D*, which acts both as a receptor and as an effector of the automaton *A* corresponds to the receptors and the effectors. An infinite tape, which is divided to successive squares, each of which contains a sign corresponds to the environment of the cognitive system (see Fig. 28).

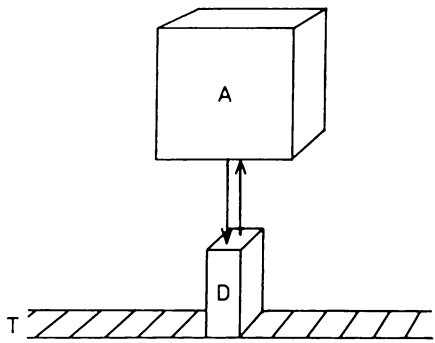


Fig. 28.

The Turing machine is also able to inspect its own output and choose its inputs, just like the cognitive system. These abilities are realized in the Turing machine in the following way: at each moment of the calendar

23. A.M. Turing, On computable numbers with an application to the Entscheidungsproblem. *Proc. London Math. Soc.*, Ser. 2, 42, p. 230—265.

of the automaton A, the device D inspects one of the squares of the tape. It first.

— reads the sign which is in the square, then

— writes a new sign (which may be the same as the old one) in the same square, and then

— either moves to the square which is next on the right side or next on the left side, or remains unmoved.

If D moves, the automation "moves its receptor". If D remains unmoved in the same square it will at the next moment of calendar time read the sign which it wrote down one moment earlier, thus "inspecting its own output". These two functions correspond to feedbacks I and II in the cognitive system.

To make things clear, let us represent the calendar of the automaton A by the set of non-negative integers:

(1) $$K = \{0,1,2,\ldots\}.$$

The machine T, by which we mean the automaton-device system, inspects at each moment t one of the squares on the tape. The external input of the machine at the time t thus is the sign a which is in the square it is inspecting at the time t:

(2) $$x^{\text{ext}}(t) = a.$$

The external output of the machine at the time t is either the order 'stop' or a combination (b, m):

(3) $$y^{\text{ext}}(t) = \text{'stop' or } (b, m).$$

The component b of the output represents the sign which is written down by the device D after it has read the sign a. The component m has three possible values m_+, m_-, and m_0. The value m_+ means that the device D moves to the square next on the right side, after it has written down the sign b, so that D will inspect the square next on the right side at the time $t+1$. The value m_- means that D moves to the square next on the left side, while the value m_0 indicates that D will stay in the same square, and will inspect the same square still at the time $t+1$. The output 'stop' means that the functioning of the machine stops: the machine will no longer read or write signs, or move along the tape.

Since A is a finite automaton, and the device D a static automaton always in the same inner state, the functioning of the machine T is completely determined by the output function f and the state-transition

function g (see p. 105). These functions determine the output and the inner state of the machine at the time $t+1$ in terms of the inner state and the input of the machine at the time t:

$$(4) \qquad \begin{cases} y^{\text{ext}}(t+1) = f(s(t), x^{\text{ext}}(t)), \\ s(t+1) = g(s(t), x^{\text{ext}}(t)). \end{cases}$$

If we denote the state $s(t)$ by s, and the state $s(t+1)$ by σ, we can rewrite this as follows:

$$(5) \qquad \begin{cases} f(s, a) = (b, m) \text{ unless it is 'stop'}, \\ g(s, a) = \sigma. \end{cases}$$

Thus the functioning of the Turing machine in the interval K_t^{t+1} is completely determined by the sequence $sabm\sigma$ (unless the machine stops).

As a finite automaton A has only a finite number, say n, of inner states: $s_1, s_2, \ldots, s_n$. Introducing a finite 'alphabet' $B = \{a_0, a_1, \ldots, a_r\}$ of the possible signs we have a Turing machine which has only a finite number of possible sequences $sabm\sigma$. Writing down all these sequences we get a *structure description* of the machine T:

$$(6) \qquad sabm\sigma;\ s'\ a'\ b'\ m'\ \sigma';\ \ldots\ldots;\ s''\ a''\ b''\ m''\ \sigma''.$$

Here each of the $s, s', \ldots, s''$ is one of the inner states $s_1, s_2, \ldots, s_m$, and each of the signs $a, a', \ldots, a''$ or $b, b', \ldots, b''$ is one of the signs $a_0, a_1, a_2 \ldots, a_r$. Each of the $m, m', \ldots, m''$ is one of the three possibilities m_+, m_-, and m_0, and each of the symbols $\sigma, \sigma', \ldots, \sigma''$ is again one of the inner states $s_1, s_2, \ldots, s_n$.

Following Turing's original paper, we can introduce a *standard description* of the Turing machine in the following way: for each s_j appearing in the above sequence as one of the symbols $s, s', \ldots, s''$ or as one of the symbols $\sigma, \sigma', \ldots, \sigma''$ we write the letter D followed by the letter A, A being repeated j times. For instance, s_3 would be written $DAAA$. For each a_j appearing in the structure description of the machine as one of the symbols $a, a', \ldots, a''$, or as one of the symbols $b, b', \ldots, b''$, we write the letter D followed by the letter C, C being repeated j times. Thus a_2, for instance, would be written CAA. For m_+ we write R, for m_- we write L, and for m_0 we write N. Then we get the standard description of the Turing machine T as a sequence composed of the symbols A, C, D, L, R, N, and the symbol ";".

We can, if we like, replace the standard description by a number. Following Turing's paper we can replace D by 1, C by 2, D by 3, L by

4, *R* by 5, *N* by 6, and ";" by 7. Then we get the complete description of the Turing machine in terms of its *description number*. This number is, of course, finite. Thus we have shown that each Turing machine can be completely characterized by a single finite number (*the Turing machines can thus be effectively enumerated*).

Example. Let the tape have in each of its squares the same sign a_0 before the machine starts working. And let the machine be such that it moves to the right all the time. Then the machine always reads a_0, writes the symbol a, and moves to the square next on the right: the machine never comes back to a square where it has already been. By such a simple Turing machine we can write down on the tape any infinite but cyclic sequence of signs we like. For instance, let us consider the machine of this type which writes down the sequence $a_1a_0a_2a_0a_1a_0a_2a_0a_1a_0\ldots$ This machine thus has an alphabet $B = \{a_0, a_1, a_2\}$. The machine needs only four inner states s_1,s_2,s_3 and s_4 and a complete description of the machine is given by

(7) $$s_1a_0a_1m_+s_2;\ s_2a_0a_0m_+s_3;\ s_3a_0a_2m_+s_4;\ s_4a_0a_0m_+s_1;.$$

The standard description is thus given by *DADDCRDAA*; *DAADDRDAAA*; *DAAADDCCRDAAAA*; *DAAAADDRDA*;. The description number of this particular Turing machine is

(8) 31332531173113353111731113322531111731111335317.

3 / Computation in the Turing Machine

The functioning of the Turing machine is called 'computation'. This is because of the original intention of Turing to represent the machine as an idealization of the process of computing as it occurs in the human mind. For instance, the inner states of the machine correspond to the mental states a human being assumes when performing computation (Turing, *ibid.*).

The Turing machine can be programmed to compute in many different ways. The way in which we make it compute also depends on the thing we want to compute. We shall consider in more detail two kinds of computation, viz. the computation of a number and the computation of a function (the values of a function) in the Turing machine.

Turing in his original paper discussed the computation of numbers only. His design for the computation of a number in the Turing machine was the following: let the tape originally have the same sign a_0 in every square. This sign is called 'blank'. Let the other signs of the finite alphabet B of the machine be called either 'figures' or 'non-figures'. We can always operate with only two figures, viz. the signs $a_1 = 0$

and $a_2 = 1$. The nonfigures $a_3, a_4, \ldots, a_r$ may be different kinds of symbols which we need not specify in more detail here. The total alphabet of our machine will thus be

$$B = \{a_0 = \text{'blank'}, a_1 = 0, a_2 = 1, a_3, a_4, \ldots, a_r\}. \tag{9}$$

The machine begins working at the initial time $t = 0$. It then reads 'blank' from a square, and writes down a figure, either 0 or 1, in this square. Then we let the machine work in such a way that the tape will at any time $t > 0$ have, when reading the tape from the initial F_1-square to the right, a sequence of signs of the general form

$$F_1 E_1 F_2 E_2 F_3 E_3 F_4 E_4 F_5 E_5 \ldots F_s a_0 a_0 a_0 a_0 a_0 \ldots . \tag{10}$$

Here each F is a figure, either 0 or 1, and each E is either 'blank' or 'non-figure'. The number s of figures written down by the machine on the "F-squares" is unlimited: we have Turing machines which will never stop. The squares having an E-sign are called "E-squares".

The sequence of figures written down by the machine before it stops,

$$F_1 F_2 F_3 F_4 F_5 \ldots F_s, \tag{11}$$

is the binary representation of the real number "computed" by the Turing machine. Of course, s is infinite only for irrational but computable real numbers. The signs on the E-squares represent signals needed for the arrangement of computation. They do not contribute anything to the final result, and they may be erased. This is Turing's original design for the computation of numbers in the Turing machine.

Computation of functions in the Turing machine can also be designed in many ways. One of these is the following: we want to compute the values of a function $F(n_1, n_2, \ldots, n_q)$, where $n_1, n_2, \ldots, n_q$ and the values of the function are non-negative integers. We can do this by using an alphabet containing only one figure, viz. the sign $a_1 = 1$. We arrange the original situation so that the tape has at the time $t = 0$ the following sequence of signs:

$$\underbrace{11\ldots1}_{n_1+1}\, a_0\, \underbrace{11\ldots1}_{n_2+1}\, a_0\, \underbrace{11\ldots1}_{n_3+1}\, a_0 \ldots\ldots \underbrace{11\ldots1}_{n_q+1}\, a_0 a_0 a_0 \ldots \tag{12}$$

Beginning with the square which the machine is reading at the time $t = 0$ there is thus n_1+1 figures 1, then a 'blank' square, then n_2+1 squares with the figure 1, then again a 'blank' square, etc., until the last n_q+1 figures 1 are followed by only 'blank' squares. The original

sequence indicated above may be written either on the successive squares, or on the alternate *F*-squares (beneath which there are the *E*-squares for operational signs as before). The construction of the machine naturally depends on the choice of writing the original sequence.

One of the designs for the computation of a function in the Turing machine is such that the machine simply writes down on the tape, after the last n_q+1 figures 1, first the 'blank' sign a_0, and then $n+1$ figures 1, there being $F(n_1, \ldots, n_q) = n$. Another design for the computation of a function is such that one counts the number n of the figures 1 on the tape when the machine stops: this number gives the value of $F(n_1, n_2, \ldots, n_q)$ in this design of computation.

Obviously computation is just another name for logical deduction. Wherever we have a mathematical or logical calculus, where some axioms and some rules of inference are given, all deduction of theorems from the axioms by means of the given rules is a problem of computation. When a Turing machine is designed so that it computes an answer, "yes" or "no", to the question whether a given theorem can be deduced from the given axioms by the given rules of inference, the Turing machine is said to be used as a *decision machine*. A theory may or may not have a decision machine.

Computation in the Turing machine may be an infinite or a finite process. It is infinite, if the machine never stops once it has started the computation. This is the case, for instance, in the computation of computable but irrational number. A Turing machine, designed to perform a certain computation, is said to be *circular* if the computation is an infinite process. If the computation stops after a finite number of steps the machine is called *circle-free*.

2 § The Turing Machine as an Idealized Model of Rational Actor

1 / The Turing Machine as a Model of the Optimal Organization of a Rational Actor

It is evident that the Turing machine can be designed to perform many kinds of computations. It can even be designed to do the same computation in many kinds of ways. In fact we have good reason to believe that with Turing machines one can compute everything which is com-

putable. This is the content of *Turing's thesis* which can be formulated as follows:

> (1) *If there is an effective procedure for the performance of some computation, there is a Turing machine in which this computation can be realized* (Turing's thesis).

The existence of an *effective procedure* or an *algorithm* means that some rules are known by means of which one can perform the calculation in successive steps in quite an unambigious way. There are finite and infinite algorithms. Of course each particular effective procedure is different from every other, and we cannot give any general definition which could include all the possible effective procedures that will ever be discovered. If we should suggest an exact definition we could never be sure that all the possible future kinds of calculation were included.

Accordingly, a statement like Turing's thesis is not, and cannot be a mathematical theorem which could be proved or disproved. It is rather a suggested 'natural law' which expresses a hypothesis on the organization of all computing systems, viz. the statement that an optimal computing system has to have the organization of the Turing machine. As soon as new kind of effective procedure for the performance of some computation is discovered we can verify or falsify the statement that this computation is realizable in a Turing machine. This has actually been done, and the verification of Turing's thesis is already rather convincing. For its generality and importance as an empirically verifiable hypothesis Turing's thesis has been compared with the two fundamental laws of thermodynamics, viz. the Conservation of Energy and the Entropy Law[24].

Why is Turing's thesis so important? Because it describes the optimal organization for a computing system. On the other hand, we know that computation in the Turing machine is equal to logical deduction. Thus, if we define the *rational actor* as a being capable of logical deduction, we can put Turing's thesis in the following form:

> (1′) *The optimal organization of a rational actor is that of the Turing machine, that is: the organization of a whole composed of a digital part, viz. a finite automaton, and some not necessarily digital receptor and effector organs so that it is able to choose*

24. See H. Hermes, *Enumerability, Decidability, Computability* (Springer 1965), p. 18. This book can be recommended to readers interested in mathematics who want more information about the Turing machines.

> *its inputs and inspect its output like a Turing machine. Such a cybernetic whole is capable of (recursive, algorithmical) logical deduction provided that its environment is rich enough in logical possibilities so that it corresponds, as a potential input store, to the infinite tape of the Turing machine.*

This more expressive, though less precise form of Turing's thesis tells roughly what a rational being should look like. Man, and especially the cognitive system of man, is an example of such a rational being. To make of man a rational being, in the sense of Turing's thesis, the nervous system or some aspect of it must be a finite automaton. This is what we must take for granted. Furthermore, man can move its receptors, and even move himself, and thus is able to choose his input. This is indicated, in the rough scheme of human organization on p. 178, by feedback I. By feedback II man is able to inspect its output. Thus, obviously man is a rational being in the sense of Turing's thesis.

We can say more than this. The phylogenetic development has gone through phases which obviously reflect a tendency toward better realization of the organization of the Turing machine. So, for instance, animals in general realize this principle better than plants. Animals have developed specialized receptors and effectors, and also the specialized central automaton, to a much greater completeness than plants. Animals are also better able to move themselves, and to move their receptors and effectors than plants. In the human being the phylogenetic development of animals has produced a being which is capable of building a great variety of artificial receptors and effectors, viz. different working tools. In this way the ability to choose the input and to control the output has improved still further from animal to man. Thus man realizes the organization of the Turing machine better than any animal. Indeed, the well known definition of man as an animal who builds working tools (Benjamin Franklin), comes very close to the definition of man as a rational being in the sense of Turing's thesis.

2 / The Universal Turing Machine as a Model of an Optimal Rational Actor

Another important statement on Turing machine is the following:

> (2) *There are Turing machines which are capable of computing everything which can be computed by any Turing machine.*

Those machines are called *universal Turing machines.* This statement, unlike Turing's thesis, is a mathematical theorem. It was proved by Turing himself. I shall sketch in the following the essential steps of Turing's proof for the existence of universal machines.

We have to construct a Turing machine U which, when given any Turing machine T_C designed to perform a certain process (C) of computation, is able to do the same computation (C). A necessary condition for this is, obviously, that one can give the machine U complete information on the machine T_C. This is possible since, as we know, the complete description of any Turing machine is a finite sequence of symbols which can be written on the tape of U before U begins working. The description of the machine T_C can thus be fed to the machine U.

The trick used by Turing for the construction of U was based on the observation that not only the machine T_C itself but also the process of computation, (C), can be completely described by a sequence of symbols suitable for printing on the tape. This sequence is finite only for a circle-free machine T_C, and infinite for a circular T_C which, however, does not matter. The first task of constructing the universal machine U is to give *a complete description of the computation* (C) for which a given Turing machine T_C is designed.

We shall restrict ourselves here, as Turing did in his original, to the discussion of only one kind of computational design, viz. to the computation of numbers as explained on pp. 182—183. As an example we shall give a simple complete description of the process of computation (C) of the Turing machine T_C which writes down on the F-squares the infinite sequence 01010101... This is the machine whose standard description and description number were constructed in the example on p. 182.

Denoting the blank squares by x the complete sequence written by T_C is $0x1x0x1x0x1x...$ Its alphabet is $B = \{a_0 = x,\ a_1 = 0\ a_2 = 1\}$, and it has four inner states s_1, s_2, s_3, and s_4. The structure description of the machine itself is (cf. p. 182)

(1) $$s_1x0m_+s_2;\ s_2xxm_+s_3;\ s_3x1m_+s_4;\ s_4xxm_+s_1;$$

It begins working in the inner state s_1, the tape being completely blank. In its first move it scans the symbol x, writes down the symbol 0, moves right, and goes over to the state s_2. In the second move it scans x, writes down x, moves right, and goes over to s_3. In the third move it scans x, writes down 1, moves right, and goes over to s_4. In the fourth move it scans x, writes down x, moves right, and goes over to s_1 back,

so that the working continues repeating these four moves cyclically. This is the process (C) we ought to describe by a sequence of symbols. We can choose, for instance, the following complete description of (C):

(2) b:0xc:0xxe:0x1xk:0x1xxb:0x1x0xc:0x1x0xxe:0x1x0x1xk:, etc.

Here we have denoted $s_1 = b$, $s_2 = c$ $s_3 = e$, and $s_4 = k$, for convenience. Each move of the computational process is described by the sequence of symbols between two subsequent colons. After each colon we have first written the sequence of symbols which is on the tape after this move, beginning with the first 0, and ending with the symbol that the machine is scanning at the end of the move. After this last symbol there is the symbol of the state to which the machine is transferred in this move. Then comes the next colon. Each sequence of symbols between two subsequent colons is called a *complete configuration* of the process (C). Thus each move of the process is described by such a complete configuration. Since the process (C) is infinite, so is the total sequence of symbols describing it.

For convenience we can introduce the *standard description of the computation* (C), just as we introduced the standard description of the machine T_C on p. 181. The latter was obtained by replacing, in the complete description of T_C, the state s_k by the letter D followed by k letter A, the symbol a_i by the letter D followed by i letters C, and the signs m_+, m_- and m_0 by the letters R, L and N, respectively. For the standard description of the machine T_C we thus obtained (cf. p. 182)

(3) *DADDCRDAA*; *DAADDRDAAA*; *DAAADDCCRDAAAA*; *DAAAADDRDA*; (S.D. of T_C).

For the standard description of the computation process, (C), we now obtain, by the same replacements as to the s_k and a_i:

(4) *DA*: *DCDDAA*: *DCDDDAAA*: *DCDDCCDDAAAA*: *DCDDCCDDDA*: *DCDDCCDDCDDAA*: etc. (S.D. of (C)).

Now that we have the standard descriptions of both the machine T_C and the process of computation (C), we can state precisely what we mean by the problem of constructing a Turing machine, say U', capable of performing the same computation (C) as the machine T_C. First we can set up a machine which, when given the standard description of T_C on the tape, can write down on the tape the successive complete configurations of (C). We can agree that both the S.D. of T_C and the

S.D. of (*C*) are to be located on the *F*-squares of the tape of U', so that the *E*-squares are left for computational signs. For the reading or printing of the standard descriptions the machine U' must have the letters *A*, *C*, *D*, *R*, and *N*, and the signs ":" and ";" in its alphabet. In order that the desired sequence 01010101... could be explicitly read from the tape of U' we can design the machine U' so that it writes down, between each successive pair of complete configurations of (*C*), the figures which appear in the new configuration but not in the old one. Thus U' would write down, instead of the S.D. of (*C*) given above, the following sequence:

(5) DA: 0: *DCDDAA*: *DCDDDAAA* :1: *DCDDCCDDAAAA*: *DCDDCCDDDA*: 0: *DCDDCCDDCDDAA*: etc.

Accordingly, U' should have in its alphabet, in addition to the letters and signs mentioned above, and in addition to a suitable number of computational signs, the two figures 0 and 1. Then the task of constructing a Turing machine U', capable of performing the same computatation (*C*) as the machine T_C, means the following: let initially the S.D. of T_C be given on some successive *F*-squares on the tape of the machine U', together with a number of computational signs (located either on *E*- or *F*-squares). The problem is to construct, using a deliberate finite number of computational signs, such state-transition and output functions for U' that it will write down on the successive *F*-squares of the tape the S.D. of (*C*) completed by the figures 0 and 1 of the sequence 01010101... as indicated above. This is a definite task to perform. After this has been done we have only to see to it that the S.D. of T_C can be exchanged for the S.D. of any other Turing machine, and we have constructed a universal machine *U*.

Above we roughly described the outline of the proof given by Turing. If you want to read Turing's original paper (mentioned in the footnote on p. 179) you should notice that he has two more technical tricks in his proof. First, when writing the complete configurations of (*C*), he combines several moves together by allowing the machine to perform several successive readings, printings, and moves within a single configuration (see his page 235). Secondly, he introduces some abbreviated notations for the writing of sequences of complete configurations. Turing's final rules for the construction of a universal machine *U*, given on his pages 244—246, are written by using these abbreviated notations.

The existence of universal Turing machines is essentially based on two facts. First, that every Turing machine can be completely described

by a finite sequence of symbols. Second, that once the machine exceeds a necessary minimum of structural complexity, it can be made to imitate the computation of any more complicated machine by compensating for the lack of complexity by deliberate lengthening of the process of computation.

The former fact we know from pp. 181—182 where we showed that every Turing machine can be completely described by a finite sequence of symbols, for instance, by the S.D. (standard description) or by the D.N. (description number). The latter fact needs some consideration. What, to begin with, is meant by structural complexity?

A fairly reasonable measure of the structural complexity of a Turing machine T would be the length $l(T)$ of its structure description. By length we mean the number of symbols in the sequence in question. The *primitive complexity*[25] thus defined is greater the more there are combinations of (input, state)-pairs and outputs, and of (input-state)-pairs and next states in the output and the state-transition functions f and g.

Now we can order the primitive complexities of Turing machines according to magnitude, beginning with the smallest:

(6) $$l_1 < l_2 < l_3 < l_4 < \ldots$$

Among the simplest machines there are no universal machines. But beginning with a certain level, say l', of primitive complexity the first universal machine U_1 appears. It is able to imitate all the Turing machines whatever their level of primitive complexity, for instance, a machine whose primitive complexity is 10^{10} times l'. At first glance this seems paradoxical. How is the machine U_1 able to perform the same functions as the bigger and more complex machine? Because we have no upper limit for the time, or for the number of successive computational steps, after which a universal machine must have completed a certain part of the computation. If the bigger machine needs m successive steps of computation (m calendar units of time), the machine U_1 may be allowed to use say 10^{10} times m steps of computation. Thus the complexity of structure is, as it were, compensated by lengthening the process of

25. Less primitive notions of complexity can be introduced, if we represent the finite automaton of the Turing machine by a neural network (the possibility of which was proved by McCulloch and Pitts, cf. p. 192). We can, for instance, define the functional complexity of a network as the length of the longest period of reverberation possible in the network ('reverberation' means the circulation of input/output around a circuit of feedback). This corresponds to the notion of 'order' of the network by McCulloch and Pitts.

computation. However, before this can happen we must have exceeded the threshold l' below which no universal machines appear.

When we think of material systems having the general organization of a Turing machine, we again come at the level l' of primitive complexity to the point where the first universal machine should appear. A material system whose structure as a Turing machine is described by the D.N. of the machine U_1 obviously is an *optimal rational being* which, theoretically, is capable of any possible sequence of logical deduction. However, for a real material being we can never assume the length of a process of computation performed by it to be unlimited. No material object is capable of preserving its structure and its mode of action forever. For instance, every material object is subjected to common wear and tear. Accordingly, no real material system can act as a universal Turing machine even though its structure might be that of a universal machine: it only may have the logical possibilities of acting as a universal machine. This is what is meant when speaking, for instance, of the human being as a universal Turing machine.

The structure of every universal Turing machine U is, of course, completely determined by the D.N. of this machine. There is an infinite sequence of universal Turing machines, each of them completely described by the respective description number in the infinite sequence $N_1, N_2, N_3, \ldots$ of the description umbers of universal Turing machines. They represent varying levels of structural complexity, there being $l(U) \geq l'$ for all U. Thus we can think of an infinite variety of real material systems, differing by structure and by structural complexity, but all representing optimal rational beings in the sense that they are all theoretically capable of any possible sequence of logical deduction.

Thus we can express the consequences of the theorem (2) for real material systems approximately as follows:

> (2′) *Above a certain level of structural complexity the real material beings having the organization of a Turing machine may become optimal rational beings in the sense that they are theoretically capable of performing any possible sequence of recursive logical deduction. Thus there is, in this sense, no theoretical upper limit for the intellectual capacity of these beings. There is an infinite variety of possible optimal rational beings, differing by structure and by structural complexity, their actual intellectual performance being restricted only by the material conditions (finite life span, energy consumption, available tools for work, sufficient environmental stimuli, etc.) to which they are subjected.*

From the formulations (1′), on p. 185, and (2′) given above we understand at once the great sugnificance of Turing's theory, originally formulated as a theory of computability, for subjective dialectics. Of course one has to keep in mind that this theory is not intended to be any realistic description of what actually happens in the human brain. Turing's theory explains the significance of the general organization of the cognitive system: to be a rational being of optimal organization the nervous system needs the peculiar mechanism for controlling its environment by means of self-inspecting and movable receptor-effector system (a mere finite automaton without the tape and the feedbacks of the types I and II on p. 178 has much more restricted ability, as was shown by C. Kleene). Turing's theory also tells us that this organization, above a certain level of structural complexity, is able to produce the optimal rational being in the sense that it is theoretically capable of all recursive logical reasoning — which is quite a lot. We can well understand that two important developments in cybernetic theory were directly inspired by Turing's theory.

First, Turing's theory immediately raises the question, 'what kind of material organization corresponds to the finite automaton of a Turing machine?' Can the automaton of a Turing machine be represented for instance by a material organization like that of the nervous system, i.e. by a kind of network of neurons? If so, we could make our theoretical understanding of the rational aspect of man somewhat more precise. This line of thought was followed up successfully by McCulloch and Pitts in their construction of neural networks, in terms of which every finite automaton can be represented.[26]

Secondly, one is led to ask: can we derive from Turing's theory a principle of reproduction of rational beings, similar to the actual reproduction of living beings? This line of thought was followed up by J. von Neumann in his theory of self-reproducing automaton.[27]

Let us only mention that both of the questions posed could be answered in the affirmative. Von Neumann was able to show that in a space (actually a plane) composed of adjacent cells, capable of assuming 29 different states, *localized Turing machines* (even universal machines) could be constructed, these machines being able to construct other machines of similar kind, and even machines more highly 'developed'

26. W. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophysics* 5, pp. 115—133, 1943.
27. J. Neumann, *Theory of Self-Reproducing Automata*, (edited by Arthur Burks), The University of Illinois Press, 1966.

than they themselves are. Thus the introduction of *spatial localization* widens the scope of Turing's theory in a decisive manner, and enables one to consider the logical conditions of reproduction of rational beings in space and time.

3 / Does the Recursivity of Its Operations Make the Turing Machine Intellectually Inferior to the Human Brain?

We have emphazised that existing cybernetic theory — both in the domain of objective social dialectics and in subjective dialectics — has important limitations. The most important one is the lack of a cybernetic theory of the transformation of quantities into qualities in the course of a self-generating dialectical process. This fact has consequences in the application of existing cybernetic theory to both objective and subjective dialectics. In subjective dialectics it appears in the lack of cybernetic theory dealing adequately with dialectical logic, i.e. with the problems related to the developmental processes of human consciousness.

However, nobody has proven that this would be a *limitation in principle*, valid for ever and for all future forms of cybernetic theory. The field is open so far for hopes that future theory will bring a solution to this problem. So, for instance, the famous mathematician and prominent cyberneticist A.N. Kolmogorov once said, "I belong to those extremist cyberneticists who see no fundamental limitations to the problem of life in the cybernetic approach and who believe it is possible to analyse life in all its aspects, including human consciousness, by cybernetic methods".[28]

An opposite opinion was expressed by the Finnish philosopher J. Hintikka, as a criticism of the Finnish predecessor of the present book. Hintikka sees in the *recursivity* of the operations of the Turing machine a limitation in principle of cybernetics to cope with all the aspects of life and human behaviour: "I have stated that the arguments given by A. in favour of the special position of cybernetics as a general method [in the mathematical methodology] of behavioral science in principle are erroneous, since they do not see the limitations of recursive methods"[29].

28. A.N. Kolmogorov, Automatic machines and the life process, *The Soviet Review*, July 1962, p. 41.
29. J. Hintikka, an official statement to the faculty of political sciences, *University of Helsinki*, 1970. Cf. with J. Hintikka, *Sosiologia* 4/1970 (in Finnish).

As the recursivity of the operations of the Turing machine is an important principle in the mathematical representation of conscious human activity, we have all reason to discuss Hintikka's argument in some detail. Is the recursivity indeed a limitation in principle, which restricts the possibilities of cybernetics in the representation of human activity, as suggested by Hintikka?

It is well known that in mathematics there are non-computable numbers and non-recursive functions which cannot be computed by means of any existing algorithm, and thus in this sense are outside the scope of recursive methods. In defence of his methodological position, Hintikka takes recourse to this trivial fact (and to the incompleteness of recursive calculi cf. p. 17). However, it is well known that the mentioned fact is not a limitation of the Turing machine in comparison with the human being: *it is the very idea of Turing's thesis that the intellectual capacity of human beings has just the same limitation to recursivity as the Turing machine.*

Hintikka is by no means alone when giving the mentioned oversimplified philosophical interpretation of the well-known results concerning recursivity and incompleteness. That interpretation has been ghosting all the time since the results in question were obtained by Gödel and Church. Orginally, it was expressed by E. Nagel and J. R. Newman in their book entitled '*Gödel's Proof*'. Recently, even the Russian mathematician A. A. Markov repeats the same interpretation at the end of his book '*The Theory of Algorithms*'.

The philosophical situation, as I see it, has been correctly represented, for instance, by Georg Klaus. He expresses the same idea by referring to the Church theorem of non-decidability: "Our epistemologically important result is thus: There are — as expressed by the Church theorem of non-decidability — sets of problems, for which no algorithm exists, in terms of which all the problems of this set could be solved. Accordingly, sets of problems, which cannot be generally solved by any electronic computer however complicated. However, this is a very weak consolation for him, who would like to see a difference of ability in principle between the human brain and the computer, as the limits of this theorem of non-decidability concern as well the brain as the computer".[30]

Klaus then refers to the obvious fact that it is man who discovers

30. G. Klaus, *Moderne Logik*, Berlin 1972, pp. 329—330.

the algorithms fed to the existing computers, which thus gives man a position over the machine. However, he adds, "But now the arguments here presented in favour of the superiority of the human brain are not of principal kind. They are associated only with the machines known today. We know that modern cybernetics has developed conceptions, which make the phenomenon of adaptation and of the improvement of adaptation understandable, i.e. the progressive development of living beings. — Starting with the fact that the human brain itself is one — and indeed the highest — form of adaptation to reality, so there are no limits in principle for the construction of machines, which cannot only technically realize the algorithms but even find them".[31]

In another book, Klaus returns to the problem of creativity, distinguishing between schematic and creative thinking, whereby the former is idealized by the algorithms realizable in a Turing machine and the latter by the probabilistic way of functioning of "statistical machines"[32].

However, here Klaus is obviously thinking of the "Turing machine" as a machine whose initial tape information (the signs on the tape before the functioning of the machine) is given once for all, and would classify a Turing machine with a probabilistic varying tape information to the "statistical machines". To avoid misunderstanding one must emphasize that the *active agent* is in both cases the same, viz. the finite automaton of the Turing machine. Only the initial conditions and the environment represented by the tape are different in the different cases: in the case of "schematic thinking" the Turing machine performs a single computation programmed for it, while in the case of "creative thinking" the same machine is located in a probabilistic environment (cf. the ideas of von Neumann, p. 172) and produces varying performances reprogramming itself according to the varying "needs" settled by the environment.

It is important to realize that[33]

> *no one has ever presented any evidence to the effect that the recursivity of the performance of the Turing machine would make this idealized machine, as a model of the formal-logical aspects*

31. G. Klaus, *ibid.*, p. 330.

32. G. Klaus, *Kybernetik und Erkenntnistheorie*, Berlin 1972, p. 258.

33. This fact has been emphasized more than once in cybernetic litterature even elsewhere. I completely agree on this issue, for instance, with M. A. Arbib, who in his book *Brains, Machines, and Mathematics* (McGraw Hill 1964) criticizes the well-known arguments of Nagel and Newman.

of rational mental activity, in principle inferior to the performance of the human brain, neither in schematic nor in 'creative' kinds of action.

Hereby it is implied

1° that the external information fed into the machine through the tape may be either once for all given (the 'schematic' performance) or even statistically variable (the 'statistical machine'), and

2° that 'in principle' in the above sentence means neglecting of the possible limitations of time, energy and material that may exist in reality.

The real difference between the machine and the brain is not to be sought in recursivity or nonrecursivity. The real difference is, firstly, in the fact that the brain has no unlimited life-span, and thus no infinite tape: no real material system has. The brain has to compensate for this deficiency, as well as possible, by the complexity of the structure of its automaton A, which makes the brain — we may well use this term here — qualitatively differing from the idealized machine. Secondly, there are all the other differences due to the fact that every real material being is subjected to material conditions, due to the consumption of energy, which are neglected in purely cybernetic theory. Thirdly, Turing's theory is lacking the theory of self-steering as yet. This means that the active, purposive aspect of the brain is neglected which, of course, is a neglection of theory only, and can be removed in future.

CHAPTER V:

Cybernetic Logic of Social Development

Modern symbolic logic has achieved much in the logical study of existing mathematical structures. This achievement was possible not least for the solid semantic foundation offered by mathematics itself for formal-logical study. As a consequence of this semantic foundation problems of consistency, completeness (of axioms) and decidability (of truth) in mathematical theories could be often successfully discussed. This is mathematical logic, understood as the logic of *mathematics.*

In recent years an extension of symbolic logic is getting more and more attention, viz. modal logic with all its branches (including among other the deontic logic). Modal logic includes the formal-logical study of notions like necessity, possibility, 'Is' and 'Ought', which are closely connected with the social reality of human beings. Therefore the semantic foundation of modal logic can hardly be found elsewhere but in a correct understanding of the factors moulding our social reality. In short: modal logic is directly concerned with *social science.*

It seems to me that modal logic has been developed on the basis of linguistic idealism, which has neglected the very fundamental factors of social reality like, for instance, the appearance of goal-directed, dialectical social development. As a consequence of this neglection the existing modal logic has no solid semantic foundation. Nevertheless, modal logic as it stands is being represented as the true logical foundation of social science, thus giving to positivistic conceptions of social science a supposed consecration in terms of 'modern logic'.

In the present chapter I shall present my criticism of the semantic foundation of existing modal logic, and study the semantic foundation of modal logic on the basis of developmental theory.

We can begin by a review of cybernetic notions related to objective dialectics, as represented by the GDR collective of philosophers in the book entitled *Marxistische Philosophie* (Dietz Verlag, Berlin 1967). This book will be referred to as *MPhi* in the following. I think that

their definitions related to the system-character of the world give a good verbal introduction even for a cybernetic approach to modal logic.

1 § The System-Character of the World

1 / Universal Causal Determinism

To make the argument explicite, it will be arranged in a tree of successive decisions, each of which is a philosophically significant choice from two alternatives:

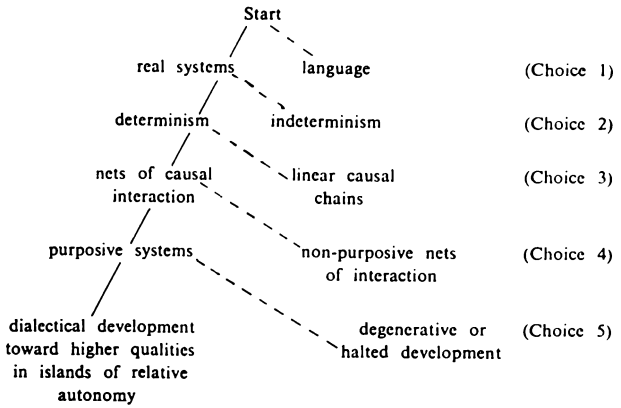


Fig. 29.

Choice 1 is one between materialism and idealism. Instead of beginning our logical study by listening to how words are used in the language (which is a customary beginning of a Western scholar of modal logic), we start from the logic of real material systems. What does it mean?

"The system character of the objective reality means that the world, (i.e.) the nature and the society, is a whole of material systems characterized by definite structural forms and being in definite relations to one another" (*MPhi*, p. 264). By virtue of this definition we can conclude that the world is thought of as a class $K = \{S_\sigma;\ \sigma \in \Sigma\}$ of material systems S_σ, each of which is characterized by a structural form

(1) $\mathrm{Str}(S_\sigma)$,

there being definite relations

(2) $\mathrm{Rel}(S_\alpha, S_\beta)$, $\mathrm{Rel}(S_\alpha, S_\beta, S_\gamma)$, etc.

between the systems $S_\sigma \in K$.

"The material systems can be divided to static and dynamic ones. However, this division eventually is only a division originating from practical ends. There are no static systems in the strict sense of the world, they are just borderline cases of dynamic systems. This statement is nothing but the system theoretical formulation of the general thesis of dialectical materialism that, in the last analysis, all rest states of systems, all unchanging systems are that only apparently or temporarily and only approximatively" (*MPhi*, p. 219). Let the systems $S_\sigma \in K$ thus be dynamic.

"Elements of material dynamic systems are used to be called active elements. They are characterized by the fact that they receive and exert influences. . . In the jargon of cybernetics the influences received by the element are called its input, the influences it exerts on the others its output. The art and manner by which the active elements of a material system are coupled with one another and with the environment of the system forms the structure of the system" (*MPhi*, p. 219—220). Thus the material dynamic systems here meant correspond, in our terminology, to the *cybernetic wholes* which simultaneously are cybernetic systems (see § 3, Chapter II). Let $E_1^\sigma, \ldots, E_N^\sigma$ be the active elements of the system (or the whole) S_σ. We can associate with S_σ the set $I(S_\sigma)$ of its elements, and write:

(3) $$I(S_\sigma) = \{E_1^\sigma, \ldots, E_N^\sigma\},$$

(4) $$\mathrm{Str}(S_\sigma) = \mathrm{Coupl}(E_1^\sigma, \ldots, E_N^\sigma, \overline{S}_\sigma).$$

Here $\overline{S}_\sigma$ means the environment of the system S_σ (i.e. $\overline{S}_\sigma$ comprehends all the material systems outside of S_σ).

Accordingly, we specify the system S_σ by the set of its active elements, and define the structural form $\mathrm{Str}(S_\sigma)$ as the mutual coupling of the elements $E_1^\sigma, \ldots, E_N^\sigma$ and the system $\overline{S}_\sigma$. By 'coupling' we understand here, generally, a statement telling us how the outputs of the elements $E_1^\sigma, \ldots, E_N^\sigma$ and of the system $\overline{S}_\sigma$ influence the inputs of these elements and the system $\overline{S}_\sigma$ (here, of course, by the output of $\overline{S}_\sigma$ we mean the input of S_σ, and *vice versa*).

After the above specification of the systems $S_\sigma \in K$ and their structural forms, the relations (2) can be specified to mean the following, using the self-explaining notation $S_\alpha \cup S_\beta$:

$$\text{(5)} \qquad \begin{aligned} &\mathrm{Rel}(S_\alpha, S_\beta) = \mathrm{Str}(S_\alpha \cup S_\beta), \\ &\mathrm{Rel}(S_\alpha, S_\beta, S_\gamma) = \mathrm{Str}(S_\alpha \cup S_\beta \cup S_\gamma), \quad \text{etc.} \end{aligned}$$

Choice 2 introduces causality as a universal principle governing all material world:

"The conception that all phenomena of the material world, on the basis of objectively effecting and recognizable laws, are in a general causal interrelation with one another and are conditioned by each other we call determinism. Correspondingly, we understand by 'indeterminism' a conception which denies the objective existence of causal interrelations as well as of general and specific lawfulnesses within the distinct forms of motion of the matter" (*MPhi*, p. 264).

We introduce determinism to our description of the world by requiring that, for each element E_i^σ of each material system S_σ, a causal relation

$$\text{(6)} \qquad R_c(E_i^\sigma) \subset X^{\sigma i} \times Y^{\sigma i}$$

is explained, telling how the input of E_i^σ determines the output of E_i^σ. Thus R_c is a two-member relation defined as a subset of the product of the set $X^{\sigma i}$ of all input states of E_i^σ and of the set $Y^{\sigma i}$ of its all output states. The distinction between cause and effect, i.e. between the sets $X^{\sigma i}$ and $Y^{\sigma i}$ is not trivial. As we have emphasized before (cf. pp. 94—96 and 99—103), it is not to be done on merely formal or conventional grounds, for instance, merely on the basis of the temporal order of events. Causality is a primary category in Marxist philosophy — a category preceding that of time (i.e. time is explained in terms of causality and not *vice versa*). The distinction between cause and effect must be done as a result of experiences directly based on the part of social practice, which is relevant in each case. A definition given by Engels is as follows:

"We find not only that a certain motion follows another, but we also find out that we can call forth a certain motion by producing the conditions under which it occurs in nature, well, that we can call forth motions which do not occur in nature at all (industry), not at least in this form, and that we can give these motions a predetermined course and range. Hereby, through the activity of man, the notion of causality is grounded, the conception that a motion is the cause of another. The

activity of man gives the test of causality" (F. Engels, *Dialektik der Natur*).

Once causality is, in accordance with formula (6), explained for the elements E_i^σ of the system S_σ the coupling (4) of the elements defines the system S_σ as a *causal net*.

There are two kinds of causal nets, viz. those involving causal interaction and those involving only linear causal chains. Let us write

$$E_i^\sigma \longrightarrow E_j^\sigma \tag{7}$$

if the output of E_i^σ is able to influence the input of E_j^σ. Then, if the elements $E_1^\sigma, \ldots, E_N^\sigma$ can be so ordered that

$$E_i^\sigma \longrightarrow E_j^\sigma \text{ if, and only if } i < j, \tag{8}$$

the system S_σ composed of the elements $E_1^\sigma, \ldots, E_N^\sigma$ is said to contain only linear causal chains (obviously, such systems are capable of only mechanical reactions in the sense of our discussion in Chapter III, cf. pp. 156—157). If (8) is not true, there is at least one circuit of feedback in the system which, accordingly, represents causal interaction (cf. p. 157). This divides the class K of all material systems to two mutually disjoint classes K_{lin} and K':

$$K = K_{\text{lin}} \cup K' \,, \quad K_{\text{lin}} \cap K' = \varnothing. \tag{9}$$

Choice 3 means focusing our interest on the class K' of feedback systems. This choice too has philosophical significance. The philosophical significance of the distinction between the classes K_{lin} and K' was explained in *MPhi* as follows:

"Here we distinguish next those systems, whose elements are mutually so coupled that they form linear causal chains, from the systems whose elements are bound together by means of feedback. The structure of the first kind of systems is such that the system reacts to the influences of the environment in a linear-causal way. Such systems we find mostly in the inorganic nature. As its prototype can serve a mechanical system of mass points under the influence of external forces . . . Quite another type of systems are obtained, if the structure of the system shows feedbacks between the elements. According to the kind of feedback such a system is able to treat the impulses it receives from the environment either so that (1) certain parameters of the system remain constant or follow in their change a definite function or so that (2) the influence of the environment upon the system ever more increases and at last enforces the system out of its domain of stability . . . The systems with compensative feedback (the case (1)) are in a certain extent able to

adapt themselves to definite types of disturbances, and to mould them so that the system does not exceed the limits of stability. This type of systems appears in particular in the organic nature and in the society. The knowledge of this type of feedback makes possible to explain in a natural way the goal-directed behaviour of many such systems, which so far ever was a domain of idealistic teleology" (*MPhi*, p. 227).

Accordingly, choice 3 means opening the way for a causal explanation of the active, purposive behaviour of living beings, and of human beings and human society in particular — the way which was closed for the mechanistic materialism, concentrated on the linear causal chains which it interpreted as the fundamental manifestation of causality in nature.

Choice 4 leads our interest to the sub-class K_{erg} of goal-directed (purposive) systems among all the feedback systems $K_\sigma \in K'$ of the world:

$$K_{erg} \subset K' \ , \quad K_{erg} \neq \varnothing. \tag{10}$$

Accepting a non-empty class K_{erg} means exceeding the barrier that in current bourgeois philosophy has been usually established between the possibilities of causal explanation and the existence of purposive beings (we thus oppose the views represented, for instance, by G.H. von Wright, *Explanation and Understanding*, London 1971).

In fact, as we have seen in Chapter III, cybernetics already knows how to build goal-directed systems, displaying a well-defined sense of 'purposiveness' on a strictly causal ground. To the existence of just such systems was referred to in the last quotation. Such a system $S_\sigma \in K_{erg}$ we have called a *self-steering (ergodic) system*. In such a system the process of interaction between the elements is, within the limits of a certain domain of stability (or rather ergodicity), independent of the external disturbances of the system. Thus, when disturbed such a system is able to eliminate the disturbance and so to impose its 'own will', provided that the disturbance does not exceed the limits of the mentioned domain. We shall apply a simple model of causally determined purposiveness in §§ 2—3 below.

As to the terminology we shall employ our previous terminology, which is the same as that used by O. Lange. If the process of interaction in our system $S_\sigma \in K_{erg}$ asymptotically approaches a definite function of time, instead of a constant, we shall speak rather of ergodicity instead of stability. A stable causal net obviously is a special case of the more interesting ergodic causal nets.

The larger is the domain of ergodicity, the more independent the system is of external disturbances or, expressed in still other words, the greater is the *relative autonomy* of the system with respect to the environment. The magnitude of relative autonomy is determined by the structural form $\mathrm{Str}(S_\sigma)$ of the system $S_\sigma \in K_{erg}$ in question, i.e. by the coupling of the elements. To be more specific, it is determined by the kind and complexity of the compensative feedbacks existing in the coupling net.

2 / Development in Multi-Ergodic Systems

Choice 5, finally, brings into our world the notion of development. "The material world, as it displays to us today in the most different domains, is a system of distinguishable stages of development of matter. To understand and to change the world we thus have to comprehend it not only in its lawful order but also in its change and development" (*MPhi*, p. 357).

Choice 5 means assuming the existence of a non-empty class

$$K_{merg} \subset K_{erg} \ , \quad K_{merg} \neq \varnothing, \tag{11}$$

of systems displaying great relative autonomy with respect to their environments, and characterized by dialectical development toward ever higher ontological qualities. What are the 'qualities', what means 'higher', and what is 'dialectical development'?

As to the qualities in the ontological sense here used: "The material world is a system of qualitatively different stages of development, which in a rough classification can be indicated as the inorganic, organic and the social stage of development" (*MPhi*, p.364). In a more specific classification of qualities the quality (this notion of ontological quality should not be confused with the notions of input and output qualities studied in Chapter II) appears as a predicate of every being and system: "The manifesting essence of beings, systems etc. we call their 'quality'. Otherwise expressed: the quality is the essential characteristic" (*MPhi*, p. 375).

On the other hand: "Every quality is quantitatively determined. A change of the quantitative determination of a being or of a system leaves, within certain limits, the quality of the being or of the system untouched. The measure (das Mass) gives the limit until which a given quality can change, without ceasing to be just this quality. By way of the notion of measure dialectical materialism thus orientates itself

to the fact that every quality is quantitatively determined and the quality and the quantity form a unity" (*MPhi*, p. 379). More precisely: "Cybernetics deals, among other, with the qualities of beings, in so far as they are measures. This makes possible to give a concrete and practical content to the general statement of Marxist philosophy, that quality and quantity form a unity within a given measure, as it has introduced a computable equivalent of measure, viz. the notion of stability or the domain of stability" (*ibid.*, p. 381). "The fact that quantitative changes within a definite quality lead, when the measure is exceeded, to the transformation of this quality to another one, we call 'the law of the transformation of quantitative changes to qualitative one' "(*ibid.*, p.382).

We conclude: the 'quality' of a system means the essential characteristic of its structure and functioning, a measure of the quality being the domain of ergodicity. In particular, the quality of an ergodic system $S_\sigma \in K_{erg}$ is changed if, and only if the system is brought out of its domain of ergodicity. This corresponds to the transformation of quantitative changes to qualitative one. Quantitative changes within the domain of ergodicity leave the quality of the system untouched.[34]

As to the general notion of dialectical development, we can here precise our earlier introduction (pp. 141—143) by quotations from *MPhi:* "The transformation of a quality to another we call a 'dialectical jump' . . . the dialectical jumps can be divided, according to the kind of dependence of the qualitative change on the preceding quantitative changes, to those determined linear-causally and to those determined non-linear-causally" (*MPhi*, p. 406). "The more complicated and complex an object is, the more importance has its self-dynamics in comparison with the linear-causal dependence on the environment" (*ibid.*, p. 407). "The non-linear-causally determined transformations of quality are no more mechanically but in the dialectical sense determined" (*ibid.*, p. 410).

An important distinction between different kinds of dialectical jumps is introduced as follows: "Through a dialectical jump an object can be destroyed, a material system ruined, made unable to work. In this case we speak of a system-destroying transformation of quality. However,

34. Existing cybernetic theory has no detailled model to explain how the construction of the new quality, i.e. the construction of the new coupling scheme and the new mode of action, is established after the breaking down of the old system. This is meant when saying that there is no mathematical model for the transformation of quantities into qualities as yet. However, the criterion of the breaking down of the old system, viz. the exceeding of the limits of ergodicity can be given as is done here.

a dialectical jump can also be of the kind that the basic quality, the essence of the object, the system etc. is not touched, or even the stability or the maintenance of the basic quality is advanced. Such dialectical jumps we call system-conserving jumps" (*MPhi*, p. 412).

More precisely: "The state of affair underlying the system-conserving dialectical jumps can be explained rather clearly by way of example of multistable cybernetic systems. Such a system is able to react to external disturbances, not as a whole but letting one or some of its subsystems move to a qualitatively new behaviour. . . .Such a multistable system is for instance the human brain, which does not react to external stimulation as a whole but mobilizes, according to the kind of the offered task, the corresponding centers in the brain. The qualitative changes occurring in the partial systems of a multistable system are system-conserving jumps, as their end is to preserve the total system and its basic quality" (*MPhi*, p. 412—413).

Dialectical development, finally, is defined in terms of the system-conserving dialectical jumps: "Without any limitation, we can thus say that the development finally can advance only through such jumps of quality, which have the system-conserving character" (*MPhi*, p.413). "By dialectical negation we understand a system-conserving transformation of quality" (*ibid.*, p. 416).

The newly introduced notion of dialectical negation leads us to the meaning of the development towards "higher qualities': "By a positive difference of development we understand the result of dialectical negations, which lead to the growth of the stability and the autonomy of the system; correspondingly, by a negative difference of development we understand the result of dialectical negations, which leads to the decrease of the stability and the autonomy of the system, If the difference of development is positive, we speak of a development forwards (Höherentwicklung) or of a progressive development; if the difference of development is negative, we speak of a development backwards (Rückentwicklung) or also of regressive development" (*MPhi*, pp. 424—425).

We conclude: dialectical materialism asserts the existence of the "islands of development" in the world, i.e. of systems $S_\sigma \in K_{\text{merg}}$ of the *multistable*, or rather *multi-ergodic* type. Such systems are characterised by development, through successive system-conserving dialectical jumps, towards ever greater ergodicity and autonomy with respect to their environments. Each of those jumps occurs in a certain part of the total system S_σ, and is system-destroying from the point of view

of the subsystem: it occurs as a consequence of the accumulation of disturbances in that subsystem, whereby the limits of the domain of ergodicity of the subsystem are exceeded so that a qualitative change (of structure and function) is called forth in the subsystem. The development occurs on strictly causal basis, and is due solely to the structure $\mathrm{Str}(S_\sigma)$ of the multi-ergodic system in question.

An important system of the multi-ergodic type is the total social system of mankind: "The proletarian revolution for instance is a system-destroying transformation of quality, which leads to the collapse of the capitalistic production relations. However, the capitalistic system is a historically determined subsystem of the total system of 'human society', which as a whole is only strengthened by the mentioned transformation of quality, and achieves a higher stage of development. The transformation of quality, which is system-destroying for the subsystem, thus is system-conserving for the total system" (*MPhi*, p. 413).

Let it be noted that we have not excluded probability from our world. As probabilistic laws can be considered as laws produced by the co-operation of a large number of non-dominating causal factors, the above construction allows the appearance of probabilistic laws just as well as it is based on the universality of strict causal determinism. However, the notion of chance, such as it appears in Marxist philosophy, has a more profound content to be studied more closely in the following.

2 § The Notions of Necessity, Possibility, Chance and Freedom

1 / A Tangential Model of Development

"The tendency that growing autonomy leads to a development of systems, independent of the quantitative and qualitative changes of the environment, that the transformations of quality are no more simply linear-causally determined enters a new stage with the birth of human society. When the plants and animals are adapting themselves, by means of complicated mechanisms of regulation, to the environment, so their autonomy is in so far restricted that they are not able to create their conditions of life, their environment. The human society, on the other hand, includes to its system the environment itself in an increasing extent in the course of historical development . . . Thereby are the causes

of development accumulated almost exhaustingly into the system of human society" (*MPhi*, p. 409).

In view of this tendency of development it is not too unrealistic to represent the total human world (see Introduction p. 8), by a closed system S, whose structure is reduced to the coupling of the active elements with one another:

(12) $\quad \mathrm{Str}(S) = \mathrm{Coupl}(E_1, \ldots, E_N).$

Let the time be indicated by successive integers,

(13) $\quad t = \ldots, -2, -1, 0, 1, 2, \ldots,$

let the sets of the thinkable total input and output states of S at the moment t be

(14) $\quad X_t \subset R^m, \quad Y_t \subset R^n,$

respectively, and let the causality and the coupling structure in the system S be defined by the respective functions

(15) $\quad T\colon X_t \rightarrow Y_{t+1}$ (causality),

(16) $\quad C\colon Y_t \rightarrow X_t$ (coupling).

The function T thus describes, how the input states of the elements of S at the moment t are transformed to the output states of the same elements at the moment $t+1$. The coupling function C indicates, how the output states of the elements at the moment t influence the input states of the other elements. We can interpret C as representing the social and other couplings established between the elements (= households, or individuals, or institutions, or whatever subsystems) of S in the course of the earlier history of the system S. In a similar way T represents the mode of behaviour of the elements as it has been moulded in the course of the earlier history.

The limitations (13) and (14) are hardly too fatal for our purpose. We digitalize the time and let the system have m input and n output channels (distributed somehow over the N elements each element E_i having at least one input and one output channel), each thinkable input or output in each channel being represented by a real number.

The limitations (15) and (16) are essential, as they fix the structure and the functioning of our system once for all. Due to the function character of T and C, involved in the assumptions (15) and (16), the model system S is not able to change its structure or its way of func-

tioning: it may be ergodic but not multi-ergodic. It follows that our system S cannot improve its ergodicity, while the real system of mankind can do that by qualitative changes in subsystems. S can model the real system of mankind only temporarily, in a certain interval of time between two successive qualitative changes of the latter. That is why we call S a tangential model.

By virtue of (15) and (16) we get two equivalent descriptions of the process of interaction occurring in S:

(17) $\varphi = C \circ T$: $X_t \longrightarrow X_{t+1}$ (the input process),

(18) $\psi = T \circ C$: $Y_t \longrightarrow Y_{t+1}$ (the output process).

Choosing the former and defining the compositions

(19) $\varphi^2 = \varphi \circ \varphi$, $\varphi^3 = \varphi \circ \varphi \circ \varphi$, etc.

we have

(20) $x(t+k) = \varphi^k(x(t))$.

Let there be

(21) $D_t \subset X_t$, $D_t \neq \varnothing$.

such that

(22) $x(t) \in D_t$ iff $\varphi^k(x(t)) \longrightarrow g_t(t+k)$ when $k \longrightarrow \infty$.

Here g_t is a fixed function of time $t+k$ independent of the initial value $x(t)$. Then D_t is the domain of ergodicity of our system S at the moment t.

A multi-ergodic system can be modelled by an ergodic system only within a certain interval of time. In our digital time, let it be the moment $t = z$ in the environment of which the processes of interaction in the ergodic model system S and in the multiergodic real system of mankind approach one another. Then we can say that the real system of mankind is at the moment z developing towards the goal represented by the direction or *goal function* g_z, the domain of ergodicity of the real system too being D_z. Soon after the moment z the development of the real system may depart from that of the model. The real system may develop, as a consequence of disturbances exceeding the domain of ergodicity of some subsystem, a qualitative change in some of its subsystems. This in turn may lead to an extension of the domain of ergodicity of the total system. Let us keep in mind this difference between our model S and the real system of mankind later, when applying the model.

The objective laws of development of human society are represented by the function φ (or ψ) in our model. What does the existence of such laws mean? It means that there are certain conditions, moulded in the course of the earlier history of humanity, and appearing at the moment z under discussion now as necessities which limit the possible ways of development of the social system — and the freedom of action of the individuals — in a definite way. Let us now study these conditions in more detail.

2 / The Semantics of Logical Modalities

Let us now consider the situation of our model at the moment z, when the model is supposed to give a first approximation of the real social system of mankind. Choosing the φ-representation of laws, the set $X_z = X$ gives the set of all thinkable states of the system S (in the following, when referring to the moment z, we shall leave off the time index z). In Marxist terminology, every element $x \in X$ then represents a thinkable state of motion of mankind, the inner law of motion of mankind being represented by the function φ. For the sake of simplicity, let us assume henceforth that the set X is finite (without this simplification we should have to speak in the following of the elements of Borel fields of subsets of X and of their σ-algebras instead of the subsets of X).

Let us generate a language L_o of sentences p in the following way. Let p_A be the sentence of L_o claiming that the world is in a state belonging to the subset A of all the thinkable states, $A \subset X$. The logical operations of negation, conjunction, disjunction and material implication are then defined by

$$(24) \qquad \begin{cases} \sim p_A = p_{X-A}, \quad p_A \,\&\, p_B = p_{A \cap B}, \quad p_A \vee p_B = p_{A \cup B}, \\ (p_A \Rightarrow p_B) = p_{X-[A \cap (X-B)]} \text{ for every } A, B, \ldots \subset X. \end{cases}$$

Here $X-A$ means the complement of A in the set X.

Let the system of mankind, where $\hat{x}$ is the real state, be denoted by $S_{\hat{x}}$. For every element $\hat{x} \in X$ there is thus a 'thinkable world' S_x which can be discussed in the language L_o. These 'worlds' are the semantic models of L_o. The total set of semantic models of L_o, or of the 'thinkable worlds', is given by

$$(25) \qquad \{S_{\hat{x}};\ \hat{x} \in X\} = SM(L_o).$$

We define the truth values W(true) and F(false) for the sentences of L_o as follows:

(26) $\quad \text{Wert}(p_A, S_{\hat{x}}) = W \text{ iff } \hat{x} \in A,$

(27) $\quad \text{Wert}(p_A, S_{\hat{x}}) = F \text{ iff } \hat{x} \in X—A.$

We define the general validity $\text{ag}_o(p)$ of a sentence p, the realizability $\text{ef}_o(p)$ of p, and the semantic equivalence $p \text{ seme q}_o\ q$ of two sentences p and q in L_o as follows:

(28) $\quad \text{ag}_o(p) \text{ iff } \text{Wert}(p, S_{\hat{x}}) = W \text{ for all } \hat{x} \in X,$

(29) $\quad \text{ef}_o(p) \text{ iff } \text{Wert}(p, S_{\hat{x}}) = W \text{ for at least one } \hat{x} \in X.$

(30) $\quad p \text{ semeq}_o\ q \text{ iff } \text{Wert}(p, S_{\hat{x}}) = \text{Wert}(q, S_{\hat{x}}) \text{ for all } \hat{x} \in X.$

The symbols ag and ef here come from the German words 'allgemeingültig' and 'erfüllbar' used by the GDR school of logicians.

It follows from the above definitions (24)—(30) that all the axioms and theorems of ordinary propositional logic — all the logical truths of this logic — are generally valid in L_o, i.e. semantically equivalent to the sentence p_X. Similarly, each proposition p_A for a non-empty A is realizable, and every sentence of L_o is semantically equivalent to some sentence p_B with some $B \subset X$. Logical contradictions are sentences which are semantically equivalent to $p_\varnothing$, where $\varnothing$ is the empty set.

So we have shown above how to describe our tangential developmental model in terms of ordinary, classical logic. However, such a description completely misses the essential point of the model, viz. the goal-directedness of the process displayed by it. To do full justice to this basic property we must construct a non-classical logic.

We must take into account that the laws of development restrict the possible states of mankind to those belonging to the domain of ergocity D. This means that not all of the thinkable semantic worlds are possible but possible are only those $S_{\hat{x}}$ for which $\hat{x} \in D$. Thus, instead of $SM(L_o)$ we have to consider the following set of semantic models:

(31) $\quad \{S_{\hat{x}};\ \hat{x} \in D\} = SM(L).$

The condition $\hat{x} \in D$ appears as a *necessity* dictated by the objective laws of development. Its character of necessity involves that human beings have no absolute freedom to choose any thinkable way of development $\hat{x} \in X$, but they are bound to go some of the ways of development inside the domain of ergodicity. Corresponding to this necessity

there is the *impossibility* of the inner states of motion $\hat{x} \in X—D$, incompatible with the laws of development.

To express the modal category of necessity we can construct the new language L preserving the sentences, the axioms and the rules of inference of L_o but giving a new semantic interpretation to the sentences of L_o. Instead of the two semantic values true and false we introduce four of them: N (necessity), U (impossibility), M (possibility), and M' (contingens).

The statement that $\hat{x} \in D$ is necessary is then expressed by

(32) $\quad \text{Wert}(p_A) = N \text{ iff } D \subset A.$

and the statement that $\hat{x} \in X—D$ is impossible by

(33) $\quad \text{Wert}(p_A) = U \text{ iff } A \subset X—D.$

Possible are all the cases which are not impossible. Thus

(34) $\quad \text{Wert}(p_A) = M \text{ iff } A \cap D \neq \varnothing.$

The semantic value M' indicates the non-necessary cases:

(35) $\quad \text{Wert}(p_A) = M' \text{ iff not } D \subset A.$

These definitions of the four semantic values can be extended to any sentence of L by requiring further that

(36) $\quad \text{Wert}(p) = \text{Wert}(q) \text{ if } p \text{ syneq } q.$

Here p syneq q means that $p \Rightarrow q$ and $q \Rightarrow p$ according to the common syntax of the languages L_o (which was constructed earlier) and L (to which it was extended by definition of L).

If $\mathfrak{N}$, $\mathfrak{A}$, $\mathfrak{M}$, and $\mathfrak{M}'$ are the classes of necessary, impossible, possible and contingent statements of L we then have:

(37) $\quad \mathfrak{M} = \mathfrak{N} \cup (\mathfrak{M} \cap \mathfrak{M}'), \mathfrak{M}' = \mathfrak{A} \cup (\mathfrak{M} \cap \mathfrak{M}').$

Here $\mathfrak{M} \cap \mathfrak{M}'$ represents the category which was by Aristotle called '*two-sided possibility*', i.e. the category of statements which are both possible and non-necessary. The total class of all the sentences of L are obviously exhausted by the following union of mutually disjoint classes:

(38) $\quad \mathfrak{T} = \mathfrak{N} \cup \mathfrak{A} \cup (\mathfrak{M} \cap \mathfrak{M}').$

In words: every proposition expressible in L is either necessary, or impossible, or then it is both possible and non-necessary.

It is the category $\mathfrak{M} \cap \mathfrak{M}'$, comprising the sentences which are both possible and non-necessary, that is close to the *Marxist category of possibility*. The existence of this logical category in a logic of development implies that, within the limits of the domain of ergodicity, the real inner state of motion of mankind is not fully determined by historical necessity. Within the limits of the domain of ergodicity there is room for chance and for the active effort of man. In other words, the domain of ergodicity indicates the limits within which the development of mankind is determined not by historical necessity but by chance and by active human pursuit.

Here *chance* is to be understood as objectively existing, i.e. as the result of numerous non-dominating causal factors (cf.p. 206) giving their contribution to the causal determination of the real state $\hat{x}$. The active human pursuit is also to be understood as an objective causal factor contributing to the causal determination of the real state $\hat{x}$.

Subjectively, the fact that human activity is able to influence the course of human history within the limits of the domain of ergodicity may be expressed as the *freedom of will*. Since the dialectical development of the real multi-ergodic system of mankind means growing ergodicity (cf. p. 205), this definition is not disagreement with the well-known definition given by Engels, and relating the increase of freedom with progressive development.

3 § The Materialistic Foundation of Modal Logic and Deontics

1 / The Logical Truths and the Dialectical Truths

The logical modalities of necessity, possibility, impossibility, and contingens were defined as characteristics of the sentences of the language L with respect to the whole set $SM(L)$ of semantic models. Picking up a particular 'possible world' $S_{\hat{x}}$ out of $SM(L)$ we can define the 'ordinary' truth values W(true) and F(false), and relate them with the semantic values N, U, M, and M'.

For a given 'possible world' $S_{\hat{x}} \in SM(L)$, $\hat{x} \in D$, we can introduce the definitions

(39) $\quad \text{Wert}(p_A, S_{\hat{x}}) = W \text{ iff } \hat{x} \in A,$

(40) $\quad \text{Wert}(p_A,\ S_{\hat{x}}) = F \text{ iff } \hat{x} \in X - A.$

The general validity ag, the realizability ef, and the semantic equivalence are thus for the sentences of the language L defined by

(41) $\quad \text{ag}(p) \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = W \text{ for all } \hat{x} \in D,$

(42) $\quad \text{ef}(p) \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = W \text{ for at least one } \hat{x} \in D,$

(43) $\quad p \text{ semeq } q \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = \text{Wert}(q,\ S_{\hat{x}}) \text{ for all } \hat{x} \in D,$

respectively.

Comparing (41)—(43) with (28)—(30) we see that for every p satisfying $\text{ag}_0(p)$ we have a fortiori $\text{ag}(p)$. Thus every logical truth of the language L_0 is a logical truth of the language L too. We also see that $p \text{ semeq}_0 q$ implies p semeq q so that every pair of semantic equivalent sentences of L_0 are semantically equivalent in L too. However, $\text{ef}_0(p)$ does not imply $\text{ef}(p)$: realizability in the language L_0 is not sufficient for realizability in L.

On the other hand, by comparing (39)—(40) with (32)—(35) we see that the modalities N, U, M, and M' are connected with the truth values W and F as follows:

(44) $\quad \text{Wert}(p) = N \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = W \text{ for all } \hat{x} \in D,$

(45) $\quad \text{Wert}(p) = U \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = F \text{ for all } \hat{x} \in D,$

(46) $\quad \text{Wert}(p) = M \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = W \text{ for at least one } \hat{x} \in D,$

(47) $\quad \text{Wert}(p) = M' \text{ iff } \text{Wert}(p,\ S_{\hat{x}}) = F \text{ for at least one } \hat{x} \in D.$

Calling the necessarily true propositions *dialectical truths* we have, by (41) and (44), the semantic equivalence of all the dialectical and logical truths:

(48) $\quad \text{if } \text{ag}(p) \text{ and } \text{Wert}(q) = N, \text{ then } p \text{ semeq } q.$

Let us think about the result (48) a little. What does it mean? The dialectical truths, i.e. the necessarily true statements in L, express the requirement that the real world must at any moment obey the objective laws of development, i.e. that the real state of motion of mankind must be within the domain of ergodicity D. The truths of formal logic are of course not outside of this requirement, as even formal logic must be considered to reflect the material reality. Thus the domain of validity of logical truths cannot be larger than the domain of validity of dialectical truths. Our result only says that, in view of this limitation,

it is maximal: the logical truths have the maximal domain of validity, and thus the same domain of validity as have the (other) dialectical truths.

2 / Strict Implication and the Modal Operators

So far we have formulated our developmental logic in terms of the language L, where the four fundamental modalities N, U, M, and M' appear as the four semantic values of the sentences of L with respect to the whole set $SM(L)$ of 'possible worlds'. Instead of this we can just as well represent the basic modalities in a language ML, where they appear as operators applying to the sentences of ML. A language in which necessity, possibility, impossibility, and contingens appear as operators is called modal logic.

The construction of the developmental modal logic ML on the basis of the developmental logic L is obvious. We have only to move all the sentences, logical connectives and axioms from L to ML, and add the new sentences Np, (for each necessarily true sentence p), Up (for each necessarily false sentence p), Mp (for each possibly true sentence p) and $M'p$ (for each possibly false sentence p). The rules of operation with the operators N, U, M, and M' can be derived from the corresponding rules of the semantic values N, U, M, and M' in L.

Obviously, all the modal operators can be expressed in terms of one of them, say, N as follows:

(49) $\qquad Up = N \sim p,\ Mp = \sim N \sim p,\ M'p = \sim Np.$

Thus the axioms and the rules of inference determining the operation with modal operators can be formulated as axioms and rules for N.

In the formal, axiomatic construction of modal logic there has been an ambiguity, which in the early history of modal logic produced some confusion. We know that in the ordinary propositional calculus all the axioms can be given the form of generally valid implications,

(50) $\qquad \mathrm{ag}(p \Rightarrow q).$

Can one take the axioms of propositional calculus such as they are, i.e. as generally valid ordinary implications, or should one construct separate axiomatics in terms of the N-implications

(51) $\qquad N(p \Rightarrow q)?$

In distinction from the ordinary 'material implication' the N-implications were called 'strict'.

Historically, the first formulations of modal logic (by C.I. Lewis) followed the latter line, while later on it was shown (by K. Gödel) that the former method could be used as well.

We can see the reason of the ambiguity associated with strict implication: all the generally valid material implications are in L, by virtue of (48), semantically equivalent to the corresponding strict implications:

(52) $\qquad \mathrm{ag}(p \Rightarrow q)$ semeq $N(p \Rightarrow q)$.

Accordingly, it is irrelevant whether the axioms of modal logic including implications are written in terms of material or strict implications.

It follows from the prededing that the axioms and the rules of inference of ordinary propositional logic can indeed be moved as such from L to ML. The additional axioms required to determine the operation with modal operators can for instance be given the following form:

1° $Np \Rightarrow p$,

2° $Np \Rightarrow NNp$ (or, in view of 1°, $N^2p = Np$),

3° $Np \Rightarrow [N(p \Rightarrow q) \Rightarrow Nq]$.

The first axiom expresses the fact that a statement of necessity is logically stronger than a statement of truth, i.e. the fact that a necessary truth is true in every possible world. The second axiom excludes, together with the axiom 1°, the artificial modal categories produced by the formal possibility of the powers of N: N^kp (and it also eliminates the mixed categories of higher order: $NMNM'UMN \ldots Np$ etc.). The third axiom is, by trivial transformations, converted to the form $Np\&Mq \Rightarrow M(p\&q)$, which is easily seen to be generally valid, in view of the semantic models $SM(L)$.

3 / The Materialistic and the Idealistic Approach in Modal Logic

We have build all the languages L_o, L and ML above way by of semantic construction. In other words, we started with the developmental model of reality S, and derived from it the 'thinkable worlds' (for L_o) and the 'possible worlds' (for L and ML) on the basis of which we constructed the languages. In such a construction the syntax of the language appears as a consequence, not as a starting point.

A semantic construction of this kind corresponds to the philosophical standpoint of materialism, according to which material reality is the primary existing thing, and the content of consciousness including language is produced by it by reflection.

Historically, modal logic was developed by means of the opposite approach. Modal logic like so much in the development of modern formal logic has been historically closely connected with philosophical idealism. Indeed modern logic as it stands is still mainly a product of mathematicians and philosophers who stand close to neopositivism, or to some more fashionable forms of philosophical idealism. It is characteristic of their approach to logic that language is for them the primary thing: at the height of the neopositivistic stream in the 1930's one even attempted the construction of a purely formal universal language of science by agreeing on some axioms and rules of inference, after which whole logic and mathematic were to be a formal play with symbols.

Now, in the 1970's, the idea of a purely syntactic construction of logic is definitely out. However, the general idea of starting with language is still there and dominates the method of Western modal logicians. This method could be called the method of *almost syntactic approach*. It consists of the attempts

1) to build modal logic and deontics by experimenting with different kinds of axiom systems, and

2) to "verify" the truth of these axioms by references to the verbal habits existing in the natural language.

The method of almost syntactic approach of logic is bound to lead to difficulties almost as surely as the neopositivistic method of purely syntactic construction. An example of this is the confusion related with the notion of strict implication (see p. 214) in the original construction of modal logical systems by C.I.Lewis. Another example is the confusion related with the first axiomatic deontic logic by E. Mally. The school of the 'almost syntactic approach' itself gives as explanation for these and other confusions met in modal and deontic logic the truism that errors have been made when applying the rules (1) or (2): lacking caution in formulation of the axioms or insufficient vigilance to the nuances of natural language. (e.g. the articles of J. Hintikka and G. von Wright in R. Hilpinen (Ed.) *Deontic Logic*, Dordrecht 1971).

From the materialistic point of view, the reason of these difficulties is more profound. It is to be found in the lack of solid semantic foundation of modal or deontic logic when based on the nuances of natural language. Instead of strained subjective introspection one should seek the foundation of modal logic and deontics in the objective laws of development of mankind.

4 / Is Many-Valued Modal Logic Needed?

Historically, both the difficulties of the first systems of modal logic (by Lewis) and of deontic logic (by Mally) were originally interpreted as suggesting the need of many-valued logic for the modal logic (suggested by J. Lukasiewitz) and for deontic logic (as suggested by K. Menger). Deontic logic will be discussed later on. We consider here only the arguments given by Lukasiewitz for the need of many-valued logic in modal logic.

Lukasiewitz, following the method of the almost syntactic approach, wanted to make modal logic by listening to the nuances of natural language. So he came to the conclusion that the statement 'if $\sim p$, then $\sim Mp$' should be a theorem of modal logic (J. Lukasiewitz, Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls, *Comptes Rendus, Soc. Sci. Letters Varsovie* XXIII, 1930, p. 51—77). He then showed that this together with the axiom $Np \Rightarrow p$ leads to logical contradiction in the two-valued logic. Another argument used by Lukasiewitz was based on the belief that if we allow the two-sided possibility, i.e. $Mp \,\&\, M \sim p$, then we should admit that everything is possible: Mq.

However, if we build modal logic on the basis of a model of development we see at once the following. The statement '$\sim p$' refers to a definite 'possible world' S_x, while the statement '$\sim Mp$' refers to the whole class $SM(L)$ of all 'possible worlds'. Thus 'if $\sim p$, then $\sim Mp$' cannot be a theorem of modal logic. In a similar way, noticing that the modal categories N, U, M, and M' refer to the total class of 'possible worlds', we see that from '$p \in \mathfrak{M} \cap \mathfrak{M}$' it does not follow that every $q \in \mathfrak{M}$.

Thus the arguments of Lukasiewitz in favour of many-valued modal logic are not correct. Of course this does not mean denying the value of many-valued logic in other contexts.

5 / Objective Ethical Value

Modal logic, when based on the objective laws of development, necessarily implies a certain deontic logic. This is seen as follows.

Let

$$\{a_\lambda(x);\ \lambda \in \Delta\ ,\ x \in X\} = a(X) \tag{53}$$

be the set of thinkable acts by human beings in the situation existing in the mankind at that moment z to which the set of thinkable inner

states of mankind $X = X_z$ refers. The argument x of $a_\lambda(x)$ is the state, $x \in X$, in which mankind is left by the act $a_\lambda(x)$. We let everything else, which characterizes the act $a_\lambda(x)$ — like the actor, for instance — be described by the index λ, as all this is irrelevant for us now. The set of all the possible acts allowed by the laws of development is then given by

$$\{a_\lambda(x);\ \lambda \in \underset{\sim}{\Delta},\ x \in D\} = a(D). \tag{54}$$

In words: possible are the acts resulting in a state of mankind which is within the domain of ergodicity of the system of mankind.

We must admit, of course, that the result x of an act $a_\lambda(x)$ is not determined by the conscious purpose of the actor only. It is determined partly even by chance, understood as an objective causal factor (cf. p. 212). However, in our present description of acts all this is included to the description of $a_\lambda(x)$ given by the index λ. Noticing this we can say that (54) expresses, in our present representation of acts, the limits of the freedom of human action existing in this particular situation of mankind.

Let $x'(t)$ and $x''(t)$ be two solutions of the inner law of motion, $x(t+1) = \varphi(x(t))$ of S, and $\xi'(t)$ and $\xi''(t)$ two state-functions of the real, multi-ergodic system of mankind, such that

$$\begin{cases} \xi'(z) = x'(z)\ , \quad \xi''(z) = x''(z), \text{ and} \\ \xi'(z+0) \text{ produces a larger domain of ergodicity than} \\ \xi''(z+0) \text{ does} \end{cases} \tag{55}$$

in the real-multi-ergodic system of mankind. Here 0 is a fixed magnitude of time. Let us rewrite (55) as a relation of weak ordering,

$$x'' \dot{\leq} x'. \tag{56}$$

We write further:

$$x'' \doteq x' \text{ iff } x'' \dot{\leq} x' \text{ and } x' \dot{\leq} x'', \tag{57}$$

$$x'' \dot{<} x' \text{ iff } x'' \dot{\leq} x' \text{ but not } x' \dot{\leq} x''. \tag{58}$$

The formula (58) reads: x' is to be preferred to x''.

The relation of weak order defined by formulae (55)—(58) determines a division of the set of possible functions $x(t)$ to a set of mutually disjoint classes of equivalence. Let this division be non-trivial, i.e. let there be at least two classes of equivalence.

Letting each function x be represented by its value at the moment z, $x = x(z)$, the formulae (55)—(58) define a preference ordering in the set $D = D_z$ of the possible states of mankind at the moment z, and thus a preference ordering in the set $a(D)$ of possible acts. Writing

$$a(x'') \dot{<} a(x') \text{ iff } x'' \dot{<} x' \tag{59}$$

we can say: the act $a(x')$ is to be preferred to the act $a(x'')$ if, and only if the state of affairs x' is to be preferred to the state of affairs x''. So we have formulated an objective system of ethical values and norms.

The procedure just outlined gives a well defined meaning for the derivation of 'Ought' from 'Is'. Indeed, surely the objective laws of development must be counted as belonging to 'Is'. It is not less sure that a statement like 'the act $a_\lambda(x')$ is to be preferred to the act $a_\lambda(x'')$' involves an 'Ought'. The only point of discussion is whether the principle of ordering, represented by formulae (55) and (56), should be called as 'derived' from the objective laws of development or whether it represents an autonomous principle of choice.

The problem is partly verbal. However, there are arguments in favour of using the word 'derivation' here. What is the content of our principle of ordering, as expressed by formulae (55) and (56)? It states that, once there are objective laws of development, imposing necessary restrictions on the pursuits of human beings, and a measure (the domain of ergodicity) for the freedom of man from these necessities, such states of affairs are to be preferred which advance the development towards greater freedom. We surely can give different formulations for the underlying principle of ordering differing from our formula (55), but they would only reflect somewhat different aspects of the objective laws of development — and thus be just as well derived from these laws. Thus I come to the conclusion that those philosophers who accept that a certain kind of 'Ought' is derivable from 'Is' have assumed a more fruitful stand as their opponents.

6 / The Modal Logic of Social Structures

The inner structure of social systems cannot be adequately represented in terms of the logical device so far discussed in this chapter. To be able to characterize properties of cybernetic couplings and elements in terms of a logical language we must move from the logic of sentences (like L_0, L, and ML) to the logic of predicates. We shall now

briefly discuss the semantic foundation of such a modal logic of social structures as based on the existence of the objective laws of human development.

Considering the developmental model S, described in § 2. 1, as a cybernetic whole its structural organization is represented by the couple (C, T) (cf. p. 121). We shall here call (C, T) the inner structure or, in short, the structure of the world S. Given (C, T), the inner law of motion of mankind φ is represented by $\varphi = C \circ T$ (cf. p. 208).

Let

$$\{(C_\sigma, T_\sigma);\ \sigma \in \Sigma\} = \mathfrak{S} \tag{60}$$

be the set of all combinations (C, T) compatible with a given total number m of input channels and n output channels. Then the set (60) represents the set of all thinkable structures of the world in an analysis where m input and n output channels are distinguished from one another.

Not all of the thinkable structures (C_σ, T_σ) are possible, i.e. realizable in view of the objective laws of development. Possible are only those structures (C_σ, T_σ) which give the true inner law of motion φ: there must be $\varphi = C_\sigma \circ T_\sigma$. Let the latter set be denoted as follows:

$$\{(C_\sigma, T_\sigma);\ \sigma \in \Sigma_\varphi\} = \mathfrak{S}_\varphi \tag{61}$$

This is thus the set composed of all the possible structures of the real world.

Every structure (C_σ, T_σ) can be produced, in the general case, by the mutual coupling of different numbers of different elementary systems E. Let the system S_σ having the structure (C_σ, T_σ) be composed of the following set of elements:

$$I_\sigma = \{E_1^\sigma, E_2^\sigma, \ldots, E_N^\sigma\}. \tag{62}$$

Every element E_l^σ is characterized by a given function

$$T_l^\sigma\colon X^{\sigma l} \to Y^{\sigma l}\ ,\quad X^{\sigma l} \subset R^{m_l}\ ,\quad Y^{\sigma l} \subset R^{n_l}, \tag{63}$$

which expresses the way of causal functioning of the element in question (cf. equation (6) on p. 146). There is $\sum_l m_l = m$, $\sum_l n_l = n$.

We define the field of individuals I of the (first-order) logic PL_1 of predicates of social structures as the class of all the elements E which can be used to produce any of the thinkable social structures:

$$I = \{E;\ E \in I_\sigma,\ \sigma \in \Sigma\}. \tag{64}$$

Thus every predicate p discussed in the language PL_1 is a characteristic of a subset I_p of a $k(p)$-fold cartesian product of I with itself: $I_p \subset I^{k(p)}$. Two predicates p and q are semantically equivalent if, and only if $I_p = I_q$. The logical connectives between given p and q are associated with the corresponding set operations between the sets I_p and I_q.

The one-member predicates then give the descriptions of any thinkable properties of the individual elements. The two-member predicates give the descriptions of any thinkable relations between any two elements, etc.

Let the set of all the elements E which appear in the construction of some or all of the systems S_σ having a structure $(C_\sigma, T_\sigma) \in \mathfrak{S}_\varphi$ be

$$I_\varphi = \{E;\ E \in I_\sigma,\ \sigma \in \Sigma_\varphi\}. \tag{65}$$

Then a one-member predicate p is necessary if, and only if $I_\varphi \subset I_p$. It is impossible if, and only if $I_p \subset I - I_\varphi$. It is possible if, and only if $I_p \cap I_\varphi \neq \varnothing$, and contingent if, and only if not $I_\varphi \subset I_p$. In a similar way, a two-member predicate p is necessary if, and only if $I_\varphi \times I_\varphi \subset I_p$, impossible if, and only if $I_p \subset I \times I - I_\varphi \times I_\varphi$, possible if, and only if $I_p \cap I_\varphi \times I_\varphi \neq \varnothing$, and contingent if, and only if not $I_\varphi \times I_\varphi \subset I_p$. Continuing in this way the modal categories N, U, M, and M' can be defined for all the predicates discussed in PL_1.

Every system S_σ having the structure (C_σ, T_σ) such that $\sigma \in \Sigma_\varphi$ defines a 'possible world'. Let $S_{\hat{\sigma}}$ be the real world. Then the set

$$\{S_{\hat{\sigma}};\ \hat{\sigma} \in \Sigma_\varphi\} = SM(PL_1) \tag{66}$$

is the set of all the semantic models of our language PL_1. The set

$$I_{\hat{\sigma}} = \{E_1^{\hat{\sigma}}, E_2^{\hat{\sigma}}, \ldots, E_N^{\hat{\sigma}}\} \tag{67}$$

is the set of the true elements of the world corresponding to the semantic model $S_{\hat{\sigma}}$. A one-member predicate p is true in the world $S_{\hat{\sigma}}$ if, and only if $I_{\hat{\sigma}} \subset I_p$, and otherwise false. A two-member predicate p is true in the world $S_{\hat{\sigma}}$ if, and only if $I_{\hat{\sigma}} \times I_{\hat{\sigma}} \subset I_p$, otherwise false. In this way the semantic values W and F can be associated, for each possible world $S_{\hat{\sigma}}$ *separately*, with the predicates discussed in PL_1.

Given a semantic model $S_{\hat{\sigma}}$ of our language PL_1 the existence of an individual E having the one-member predicate p means that $I_{\hat{\sigma}} \cap I_p \neq \varnothing$. The existence of a sequence (E_1, E_2) of individuals having the two-member predicate p means that $I_{\hat{\sigma}} \times I_{\hat{\sigma}} \cap I_p \neq \varnothing$, etc. In this way the quantifiers and the truth values of sentences including quantifiers can be defined in PL_1.

The definition of the semantic values N, U, M, and M' for sentences containing quantifiers is obvious. For instance, if the sentence $\mathrm{Ex}(E)p(E)$ is true for all the possible worlds $S_{\hat{\sigma}}$, $\hat{\sigma} \in \Sigma_{\varphi}$, i.e. for all the structures of the world allowed by the laws of development, then it is a necessary characteristic of the real world. If the sentence $\mathrm{Ex}(E)p(E)$ is true for at least one of the possible worlds, it expresses a possible characteristic of the real world, etc.

The above hints suffice to show how our cybernetic logic of development is to be extended for a logico-mathematical discussion of social structures. Such a discussion, however, and social science in general is outside of the subject of this book.

Obviously, a two-level theory of social knowledge is suggested. The empirical level, concerning the historical process $x(t)$, and the theoretical level, concerning the social structures (C, T) are connected with one another by the law of development φ.

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